Hall MS Math Festival — Dec. 1, 2023

GAMES OF LITTLE TO NO CHANCE

For all but problem #2, two players alternate turns, with the same rule for making legal turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works. And, always, try to generalize!

1 *Takeaway*. A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).

Solution: Call the starting value s; there is nothing special about $s = 17$. Examining smaller values, it is easy to see that player A wins on her first move if $1 \leq s \leq 4$. If $s = 5$, no matter what A does, B will be presented with a value between 1 and 4, inclusive, and will then win in a single move. Indeed, B is now the first player "up," and hence has the same fate that A had; i.e., winning.

In other words, presenting one's opponent with the value 5 guarantees that you (not your opponent) will win. So if $6 \leq s \leq 9$, player A will take away enough pennies to present B with 5. Continuing this analysis further, it is clear that as long as you present your opponent with a multiple of 5, you will win. Why?

- If you present your opponent with a multiple of 5, she *must* present you with a non-multiple of 5.
- Conversely, if you present your opponent with a non-multiple of 5, she *can choose to* present you with a multiple of 5.

Consequently, if s is a non-multiple of 5, then A can guarantee a win by presenting her opponent with successively smaller multiples of 5, culminating in 0.

Using coloring. We will systematically color the game positions with green for winning and red for losing, by applying two simple rules. Consider the number line starting at 0. We start by coloring 0 green, of course, since if we presented our opponent with 0, then we have won!

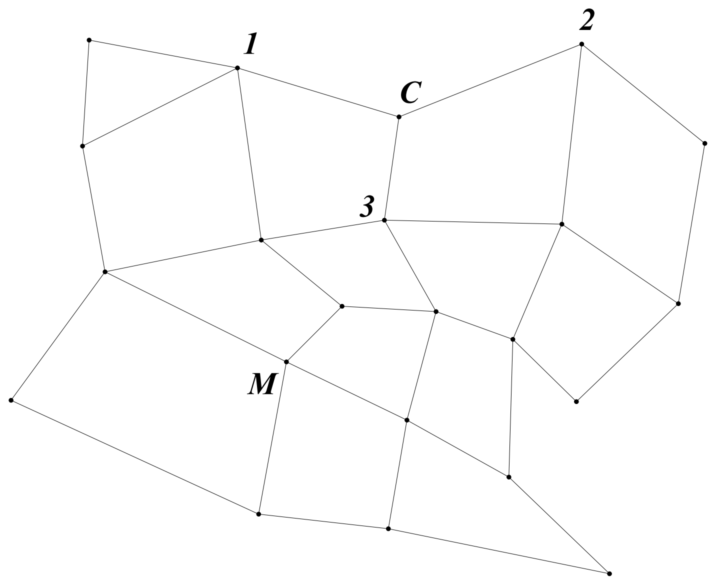
Next, moving to right on the number line, we apply the following coloring rules:

1. *Red rule*. Color a value red if, starting from this position, we can move to a green position *in one move*.
2. *Green rule*. Color a value green if there are no smaller uncolored values.

Verify that this procedure will color the multiples of 5 green, and every other value red.

The coloring method is a simple way to analyze a game without thinking too hard. It is a good way to get your hands dirty in an initial investigation.

2 *Cat and Mouse*. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?

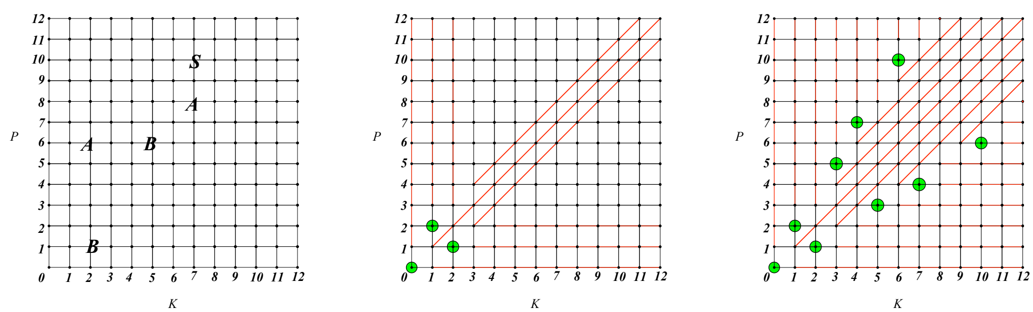
Solution: Yes, the cat wins, by first moving to point 1 and then moving to the upper left corner on its second move and then down on move #3. This way, the cat visits all three vertices of the only triangle, and thus switches the tempo. This is easier to see with coloring: notice that the game network is almost a checkerboard; if you color the cat's starting point black and the then alternate colors with white (avoiding coloring the upper-left), you will see that the cat and mouse both start at a black point. Thus, when the cat moves, it moves to the *wrong* color, and can never catch the mouse... unless it moves to the upper-left, and then down on its next move. Then you will see that the cat is moving to the correct color. Try it with coloring!

3 *Puppies and Kittens*. We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.

Solution: the game states are two-dimensional. The game can be visualized geometrically using a horizontal kitten and vertical puppy axis. The game starts at a lattice point in the plane, and legal moves are due west, south, or southwest, with the objective of reaching the origin first. The leftmost grid in the figure below illustrates a sample game. Starting with 10 puppies and 7 kittens, player A adopted 2 puppies, then B adopted 2 of each, then A took home 3 kittens, and B took 5 puppies, leaving A with the game state $(2,1)$. It should be clear that this forces a win for B.

In other words, $(2,1)$—and by symmetry, $(1,2)$—are green points, along with $(0,0)$. A moment's thought shows that the entire axes and diagonal lattice points northeast of the origin are colored red, as depicted in the middle grid (the red points are shown by lines; the green points are circles). This is what forces $(2,1)$ and $(1,2)$ to be green: they are the "first" points that *cannot* reach $(0,0)$ in one move. Instead, a player whose position is, say, $(1,2)$ has no choice but to move to a red point.

Continuing in this way, we see that there are infinitely many red points that are one move away from $(1,2)$ and $(2,1)$; namely all points north, east, and due northeast of them. A careful perusal of the graph now shows that the new green points are $(3,5)$ and $(5,3)$. Starting from either of these, any south, west, or southwest move lands us on a red point, and from there, one can chose a move to a green point, etc. Continuing this process, we can easily work out several more green points, as shown in the rightmost grid. Listing just the ones where $k \le p$, our list so far is $(0,0),(1,2),(3,5),(4,7),(6,10)$.



The graphical method, while appealing, is not necessary. For example, let's find the next green point after $(6,10)$. Since all points to the north, east, and northeast of the current green set are colored red, we have eliminated all points with coordinates used so far (i.e., all points with coordinates equal to $0,1,2,3,4,5,6,7,10$) as well as all points where the difference of the coordinates is $0,1,2,3,4$. The next green point will have a coordinate difference of 5, and cannot use any of the forbidden values. Thus it must be $(8,13)$. Using this method, we can easily compute more green points. Here is a table of the first values, where d denotes the difference of coordinates, and we just choose the points where $k \le p$ (so if, say, $(8,13)$ is green, then so is $(13,8)$, etc.).

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13
k	0	1	3	4	6	8	9	11	12	14	16	17	19	21
p	0	2	5	7	10	13	15	18	20	23	26	28	31	34

A crucial question is whether there is a closed-form formula for the green coordinates in terms of d. Fibonacci numbers seem to lurk in this table. Clearly something interesting is going on involving them.