14th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 28, 2012

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Answer the following two questions and justify your answers:

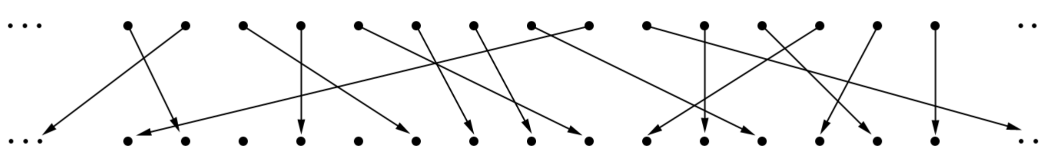
(a) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + 5^{2012}$?

(b) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + \cdots + 2011^{2012} + 2012^{2012}$?

2 Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:

- No two arrows point at the same dot.
- No arrow can extend right or left by more than 1006 positions.

Show that at most 2012 dots in the lower row could have no arrow pointing to them.



Please turn over for the remaining problems!

3 Let $x_1, x_2, \ldots, x_k$ be a sequence of integers. A rearrangement of this sequence (the numbers in the sequence listed in some other order) is called a **scramble** if no number in the new sequence is equal to the number originally in its location. For example, if the original sequence is $1,3,3,5$ then $3,5,1,3$ is a scramble, but $3,3,1,5$ is not.

A rearrangement is called a **two-two** if exactly two of the numbers in the new sequence are each exactly two more than the numbers that originally occupied those locations. For example, $3,5,1,3$ is a two-two of the sequence $1,3,3,5$ (the first two values 3 and 5 of the new sequence are exactly two more than their original values 1 and 3).

Let $n \geq 2$. Prove that the number of scrambles of

$$1,1,2,3,\ldots,n-1,n$$

is equal to the number of two-twos of

$$1,2,3,\ldots,n,n+1\ .$$

(Notice that both sequences have $n+1$ numbers, but the first one contains two 1s.)

4 Given a segment AB in the plane, choose on it a point M different from A and B. Two equilateral triangles $\triangle AMC$ and $\triangle BMD$ in the plane are constructed on the same side of segment AB. The circumcircles of the two triangles intersect in point M and another point N. (The **circumcircle** of a triangle is the circle that passes through all three of its vertices.)

(a) Prove that lines AD and BC pass through point N.

(b) Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

5 Find all nonzero polynomials $P(x)$ with integer coefficients that satisfy the following property: whenever a and b are relatively prime integers, then $P(a)$ and $P(b)$ are relatively prime as well. Prove that your answer is correct. (Two integers are **relatively prime** if they have no common prime factors. For example, -70 and 99 are relatively prime, while -70 and 15 are not relatively prime.)

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2012 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 11. This event will include lunch, a mathematical talk, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.