16th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 25, 2014

The time limit for this exam is 4 hours. Your solutions should contain clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together. Even if your solution is less than one page long, please begin each problem on a new sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won.

Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

2 Let ABC be a scalene triangle with the longest side AC. (A *scalene* triangle has sides of different lengths.) Let P and Q be the points on the side AC such that $AP = AB$ and $CQ = CB$. Thus we have a new triangle BPQ inside triangle ABC. Let k_1 be the circle *circumscribed* around the triangle BPQ (that is, the circle passing through the vertices B, P, and Q of the triangle BPQ); and let k_2 be the circle *inscribed* in triangle ABC (that is, the circle inside triangle ABC that is tangent to the three sides AB, BC, and CA). Prove that the two circles k_1 and k_2 are *concentric*, that is, they have the same center.

3 Suppose that for two real numbers x and y the following equality is true:

$$(x+\sqrt{1+x^2})(y+\sqrt{1+y^2}) = 1.$$

Find (with proof) the value of $x+y$.

Please turn over for the remaining problems!

4 Let $F_1, F_2, F_3, \ldots$ be the *Fibonacci sequence*, the sequence of positive integers satisfying

$$F_1 = F_2 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 1.$$

Does there exist an $n \geq 1$ for which F_n is divisible by 2014?

5 A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings.

It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n - \sqrt{2k}$ and no more than $n + \sqrt{2k}$ games.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2014 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11 am-2 pm on Sunday, March 19. This event will include a mathematical talk, a mathematicians' tea, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.