Dissections **Wed PM**

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The **Wallace-Bolyai-Gerwein** theorem states that any two polygons in the plane with equal area are "scissors congruent;" i.e., you can cut one polygon into pieces which can be perfectly fit together (no holes, no overlaps) to form the other. I am indebted to Lalit Jain, a high school teacher at the time, who taught me about this at a workshop in Berkeley.

1 Prove the formula for the area of a triangle in as many ways as possible, including using paper and scissors. Does your proof work for any triangle? Why does it work?

2 Why does the area of a parallelogram only depend on its height and base? Explain this in more than one way.

3 Can any polygon be dissected into triangles? Why? Have you examined both the convex and non-convex cases?

4 Show that two rectangles of equal area are scissors congruent.

5 Do the above problems allow you to prove the WBG theorem?

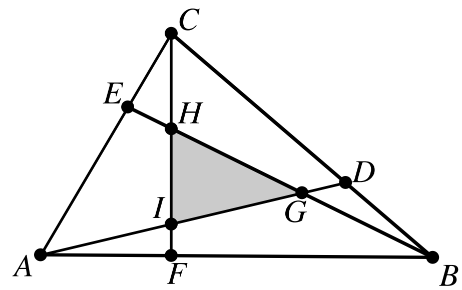
6 The WBG theorem deals with 2-dimensional polygons. What about other planar objects? And what about 3-dimensional shapes?

Other Area and Dissection Problems

7 Do you know any proofs of the Pythagorean Theorem that use dissections? What does this have to do with the WBG theorem?

8 *Medians*. We all know that the three medians of a triangle (the lines going from a vertex to the midpoint of the opposite side) intersect in a point and that the intersection point cuts the medians in a $1:2$ ratio. Accept, for the moment, that the medians meet in a point (although that is worth proving from scratch), but use your knowledge about area to deduce the $1:2$ ratio, with ease!

9 *Threedians*. The lines AD, BE, CF below are "threedians;" in other words, they hit the opposite edges at a trisection point ($CE = AC/3$, etc.). What can you say about the relationship of the shaded area $[GHI]$ to the area $[ABC]$?



10 *Infinite dissections*. (Thanks to Sam Vandervelde.) Can you dissect a square into infinitely many line segments? Of course you can. (A line segment, by the way, is straight and has two endpoints and infinitely many points in-between; in other words, it has positive, non-zero length. A single point is *not* a line segment. And "dissecting into line segments" means decomposing into *disjoint* line segments (every point of the target shape is covered by a line segment, and no two line segments have *any* point in common, and no other points are covered). So using this definition of line segment, here are a few harder questions. Which of the following can you dissect into infinitely many line segments?

(a) A rectangle.

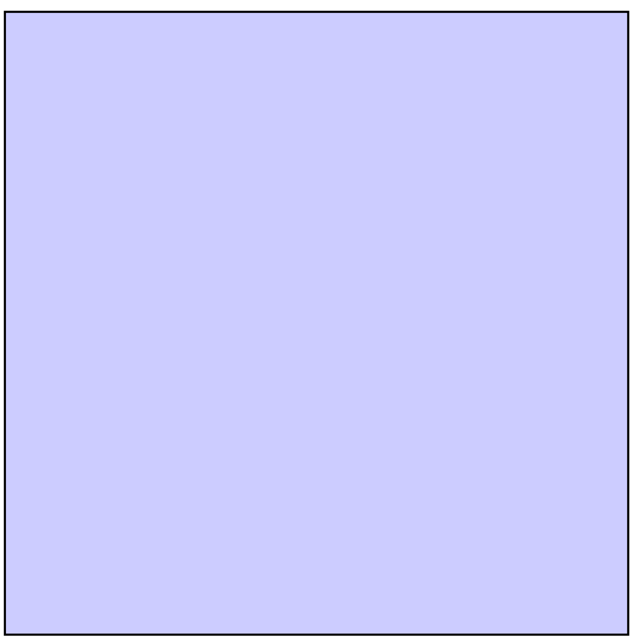
(b) A trapezoid.

(c) A triangle.

(d) A semicircle.

(e) A circle.

The two rectangles on this page have the same area, 17 square units. The top one has dimensions $\pi \times 17/\pi$; the bottom one is a square of dimensions $\sqrt{17} \times \sqrt{17}$.



Mathemtical magic for muggles **Wed AM**

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Here are several easy-to-perform feats that suggest supernatural powers such as telepathy, "seeing fingers," predicting the future, photographic memory, etc. Each trick uses simple mathematical ideas that allow information to flow effortlessly and sneakily, such as parity and symmetry.

One can approach these activities in many ways. At first, you may want to figure out HOW to do a trick. Then, you want to know WHY it works. Finally, you should strive to understand REALLY WHY it works: is there a simple theme or principle behind your possibly complex explanation? Look for simple and general guiding principles.

Several of these tricks were researched, perfected, and classroom-tested in 2012 at the San Francisco Math Circle by SFSU grad students Jessica Delgado and Kelly Walker. I am indebted to them. In turn, they (and I) are also indebted to the book *Magical Mathematics*, by Persi Diaconis and the late, great Ronald Graham (Princeton University Press, 2012).

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1 *Warm-up: Fingers That Can See*. The Magician deals cards on a table (not in a pile), placing them face up or face down on the command of the Participant, and stops dealing when the Participant says so.

Then the Magician is blindfolded. The Magician proceeds to put the cards into two piles, using their magical seeing fingers, so that, miraculously, each pile has exactly the same number of face-up cards!

2 *Zvonkin's Magic Table*. This trick is adapted from A. Zvonkin's book *Math From 3 to 7*, which I helped to translate and edit. Zvonkin ran a math circle for small kids in Moscow and entertained them by having them cover any four consecutive numbers in the table provided (vertical or horizontal), and then he would instantly determine the sum! Was it a feat of memory? Telepathy?

3 *Telepathic Teacher*. The Teacher, blindfolded, asks a Student to write a largish (four to six digits, say) number on the board. The student is instructed to then write the number backwards, and to subtract the smaller of the two from the larger, with other students quietly checking the work to make sure it is perfect.

Then the Teacher asks the student to circle one digit in the answer, and then say what the other digits are. The class is then asked to concentrate deeply on the circled digit. The Teacher is able, with high probability, to correctly name the digit.

How is this done? Why is it only with "high probability?"