Scissors Congruence

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The **Wallace-Bolyai-Gerwein** theorem states that any two polygons in the plane with equal area are "scissors congruent;" i.e., you can cut one polygon into pieces which can be perfectly fit together (no holes, no overlaps) to form the other. I am indebted to Lalit Jain, a high school teacher at the time, who taught me about this at a workshop in Berkeley.

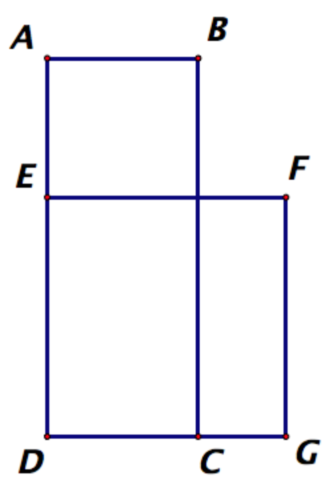
1 Prove the formula for the area of a triangle in as many ways as possible, including using paper and scissors. Does your proof work for any triangle? Why does it work?

2 Why does the area of a parallelogram only depend on its height and base? Explain this in more than one way.

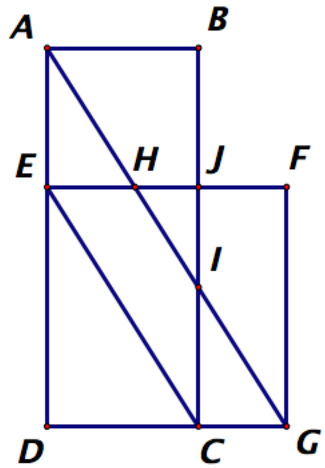
3 Can any polygon be dissected into triangles? Why? Have you examined both the convex and non-convex cases?

4 Show that two rectangles of equal area are scissors congruent.

Solution: Consider rectangles $ABCD$ and $DEFG$ below, both with the same area. I drew this in Geometer's Sketchpad so that the first rectangle was 5×2 while the second was $\sqrt{10} \times \sqrt{10}$.

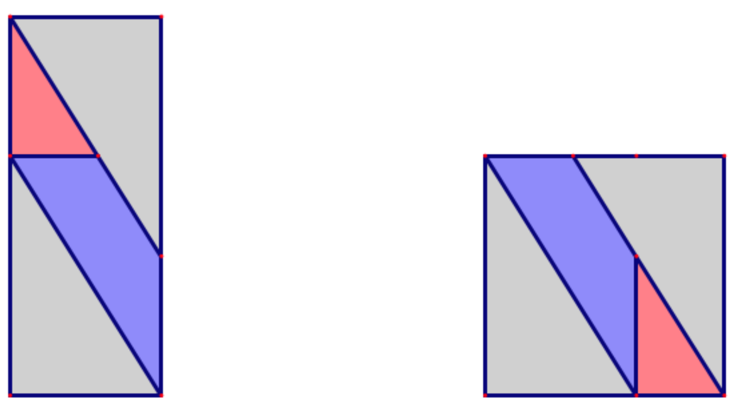


The key idea is to find some cut that "marries" both rectangles. Since the areas are equal, we have $DC \cdot AD = DG \cdot ED$, so $AD/DG = ED/DC$. This suggests *similar triangles*, and practically *demands* that you draw in the line joining A and G and the line joining E and C!



Because $EDC \sim ADG$, lines AG and EC are parallel, and there are many more similar triangles: $EJC \sim HFG \sim ABI$, and in fact, these three triangles are *congruent* because $AB = DC$ and $JC = FG$. Likewise, since $EHGC$ is a parallelogram, $EH = CG$, and triangles AEH and ICG are not just similar, but congruent.

If you think about it, this is enough to suggest a very simple dissection, requiring just two straight-line cuts!



Remark: Notice also that this method allows you to take any rectangle and dissect it to become another rectangle using any specified base. For example, you can take the 2×5 rectangle, and draw line DG to have length, say, 2π. Then we can draw line AG as before, find the intersection point I, and cut along AG and slide triangle ABI down so that I coincides with G. Then the new location of point B is the upper-right corner of the rectangle with base 2π and height $5/\pi$, preserving the area of 10. The point of this: *we can take any two rectangles and dissect them to form a bigger, single rectangle.*

One caveat: what if the the dimensions are really wacky? For example, suppose one rectangle is 1×1000000 and the other is 1000×1000. Then (verify!) you may need to modify the above method and do more translations (similar to what you may need to do with parallelograms).

5 Do the above problems allow you to prove the WBG theorem?

Solution: Yes, since any polygon can be triangulated, and each of these triangles can be turned into rectangles. And you can take any two rectangles and build a bigger rectangle (longer or taller) using the remark above. Eventually, you can dissect any polygon and turn it into a single rectangle. And likewise you can turn the other polygon into a single rectangle. You can dissect one of these giant rectangles into the other and then play the film backwards. It's not pretty, but—in theory—it can be done using only the triangle-to-parallelogram, parallelogram-to-rectangle, and rectangle-to-rectangle dissections!

Remark: The WBG theorem is false in 3 dimensions (due to Dehn), and other 2-dimensional shapes are poorly understood.

Other Dissection Problems

7 Do you know any proofs of the Pythagorean Theorem that use dissections? What does this have to do with the WBG theorem?

8 *Infinite dissections*. (Thanks to Sam Vandervelde.) Can you dissect a square into infinitely many line segments? Of course you can. (A line segment, by the way, is straight and has two endpoints and infinitely many points in-between; in other words, it has positive, non-zero length. A single point is *not* a line segment. And "dissecting into line segments" means decomposing into disjoint line segments (no two line segments have any point in common). So using this definition of line segment, here are a few harder questions. Which of the following can you dissect into infinitely many line segments?

(a) A rectangle.

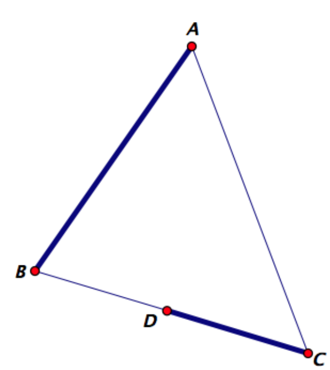
Solution: This is pretty obvious. Just draw parallel line segments.

(b) A trapezoid.

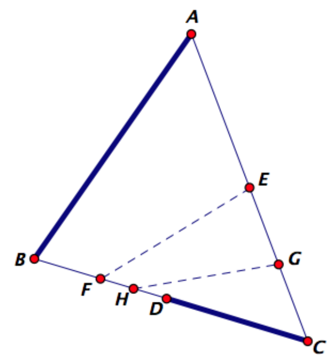
Solution: So is this.

(c) A triangle.

Solution: This is a nice example of using wishful thinking. If only a triangle were a trapezoid, say. *Make it so!*

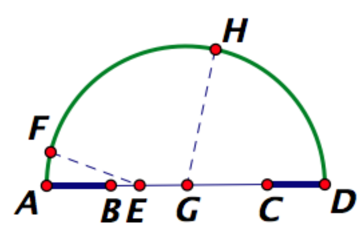


Who said that any of the sides of the trapezoid need to be parallel? Now we proceed as if we had a trapezoid. The two shaded lines AB and CD are the initial two line segments of the dissection and then we can draw line segments that include the rest of the triangle as follows: Along line BD, for example, the midpoint is F. We join F with the midpoint of AC. Likewise, point H is $3/4$ of the way from B to D, so we join it to G, which is also $3/4$ of the way between A and C. In this way, we can pick any point on BD and find a corresponding point on AC to join it to. The picture below illustrates this. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin lines are not one of the dissection line segments, but merely guidelines.)



(d) A semicircle.

Solution: We use the same idea as with the triangle, only twice. Start with segments AB and CD. Then we imagine that these two segments are the sides of a "trapezoid" whose top is the arc of the semicircle and whose bottom is BC. For example, EF and GH are two of the segments in the dissection. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin line is not one of the dissection line segments, but merely a guideline.)



If you are not satisfied with this picture, and want a formula, you can assign to each point on the interval between B and C an angle between 180 and 0 degrees, and then join each point on the BC to the corresponding point on the arc with that angle. For example, B joins with A (angle 180), and C joins with D (angle is 0) and E joins with 170 degrees, G joins with 77 degrees, etc.

(e) A circle.

Solution: Here's a picture to illustrate the idea. Essentially, we take two "semicircles" and put them together, but leave space for a "belt" of parallel lines. Again, the thick and dashed lines are line segments used in the dissection, but the thin lines are merely guidelines.