SFMTC S2.E1 171021

Scissors Congruence

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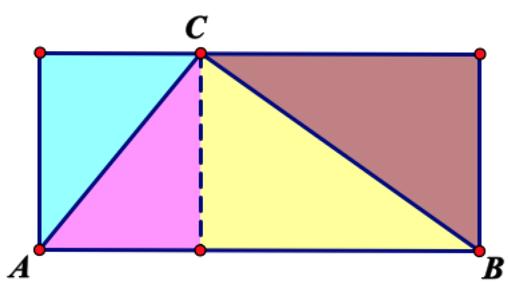
The **Wallace-Bolyai-Gerwein** theorem states that any two polygons in the plane with equal area are "scissors congruent;" i.e., you can cut one polygon into pieces which can be perfectly fit together (no holes, no overlaps) to form the other. I am indebted to Lalit Jain, a high school teacher at the time, who taught me about this at a workshop in Berkeley.

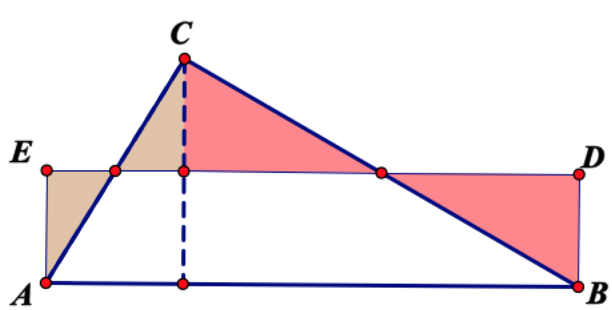
1 Prove the formula for the area of a triangle in as many ways as possible, including using paper and scissors. Does your proof work for any triangle? Why does it work?

Solution: The following picture shows that the area of triangle ABC is half the area of the enclosing rectangle, since the perpendicular dropped from C divides things into pairs of obviously congruent triangles.



But what if the triangle is really wacky, with a big "lean," and doesn't fit *inside* the rectangle? You will need to modify the above ideas.

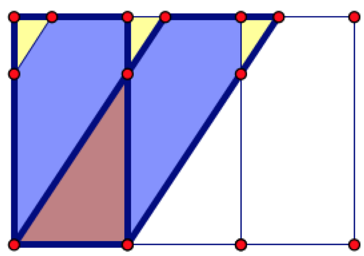
Another approach is to *directly* show that the area of a triangle is equal to half the area of a rectangle. As before, drop the perpendicular, but now draw a line parallel to the base of the triangle that goes through the midpoints of the sides of triangle.



Again, pairs of congruent triangles makes it obvious that we can dissect triangle ABC and reassemble it as rectangle $ABDE$. As before, if the triangle leans far (i.e., the perpendicular from C to AB lies outside of AB), you will need to modify your solution. How?

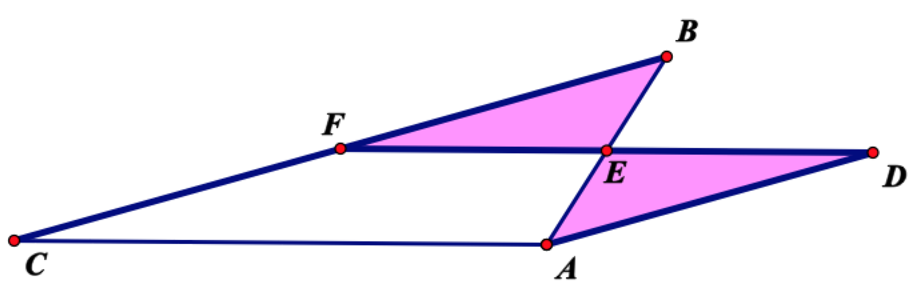
2 Why does the area of a parallelogram only depend on its height and base? Explain this in more than one way.

Solution: Certainly you can divide any parallelogram into two congruent triangles, and use the previous problem, but it's fun to see it *directly*. Study the picture below, which shows that the area of the rectangle (thick lines) is equal to the area of the parallelogram with the same base and height (again, thick lines), by drawing a few translated copies of the rectangle and watching how it interacts with the parallelogram. The regions with the same colors are conguent; why? Isn't it beautiful?



Notice that if the parallelogram leans a lot, we need to draw more translates of the rectangle. Try it!

Also, notice that once we agree that parallelograms and rectangles have the same area if they have the same base and height, we can develop another, elegant, scissors-congruence demonstration of the area of a triangle that works even if the triangle has an extreme lean. In the picture below, by drawing (as before) the line FE joining the midpoints of sides BC and AB, it should be evident that the area of triangle ABC is equal to the area of parallelogram $ABCD$.

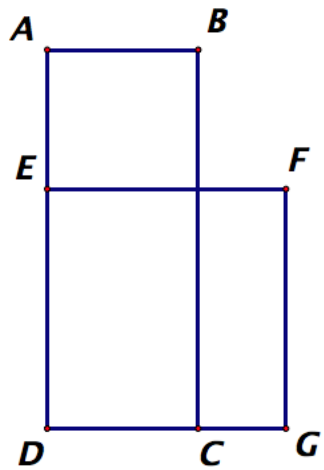


3 Can any polygon be dissected into triangles? Why? Have you examined both the convex and non-convex cases?

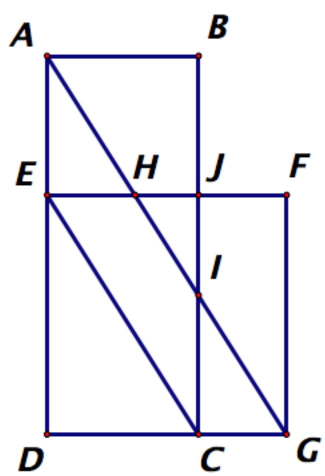
Solution: All I will say here is "yes," but a formal pick is pretty picky. But less interesting than the other problems, in my opinion.

4 Show that two rectangles of equal area are scissors congruent.

Solution: Consider rectangles $ABCD$ and $DEFG$ below, both with the same area. I drew this in Geometer's Sketchpad so that the first rectangle was 5×2 while the second was $\sqrt{10} \times \sqrt{10}$.

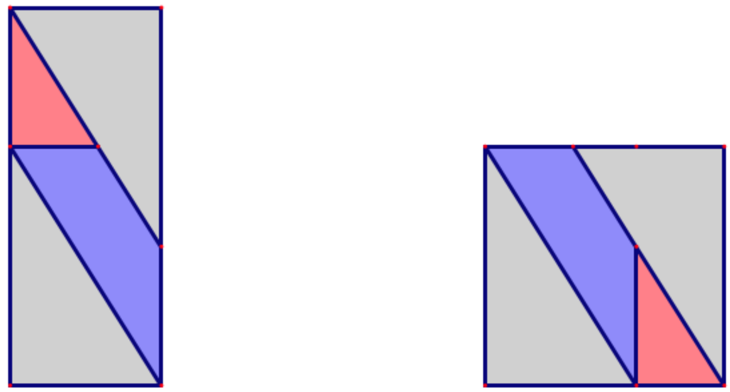


The key idea is to find some cut that "marries" both rectangles. Since the areas are equal, we have $DC \cdot AD = DG \cdot ED$, so $AD/DG = ED/DC$. This suggests *similar triangles*, and practically *demands* that you draw in the line joining A and G and the line joining E and C!



Because $EDC \sim ADG$, lines AG and EC are parallel, and there are many more similar triangles: $EJC \sim HFG \sim ABI$, and in fact, these three triangles are *congruent* because $AB = DC$ and $JC = FG$. Likewise, since $EHGC$ is a parallelogram, $EH = CG$, and triangles AEH and ICG are not just similar, but congruent.

If you think about it, this is enough to suggest a very simple dissection, requiring just two straight-line cuts!



Remark: Notice also that this method allows you to take any rectangle and dissect it to become another rectangle using any specified base. For example, you can take the 2×5 rectangle, and draw line DG to have length, say, 2π. Then we can draw line AG as before, find the intersection point I, and cut along AG and slide triangle ABI down so that I coincides with G. Then the new location of point B is the upper-right corner of the rectangle with base 2π and height $5/\pi$, preserving the area of 10. The point of this: *we can take any two rectangles and dissect them to form a bigger, single rectangle.*

One caveat: what if the the dimensions are really wacky? For example, suppose one rectangle is 1×1000000 and the other is 1000×1000. Then (verify!) you may need to modify the above method and do more translations (similar to what you may need to do with parallelograms).

5 Do the above problems allow you to prove the WBG theorem?

Solution: Yes, since any polygon can be triangulated, and each of these triangles can be turned into rectangles. And you can take any two rectangles and build a bigger rectangle (longer or taller) using the remark above. Eventually, you can dissect any polygon and turn it into a single rectangle. And likewise you can turn the other polygon into a single rectangle. You can dissect one of these giant rectangles into the other and then play the film backwards. It's not pretty, but—in theory—it can be done using only the triangle-to-parallelogram, parallelogram-to-rectangle, and rectangle-to-rectangle dissections!

6 The WBG theorem deals with 2-dimensional polygons. What about other 2-dimensional objects? And what about 3-dimensional shapes?

Remark: The WBG theorem is false in 3 dimensions (due to Dehn), and other 2-dimensional shapes are poorly understood. Google "Hilbert's 3rd Problem" to learn more.

Other Area and Dissection Problems

7 Do you know any proofs of the Pythagorean Theorem that use dissections? What does this have to do with the WBG theorem?

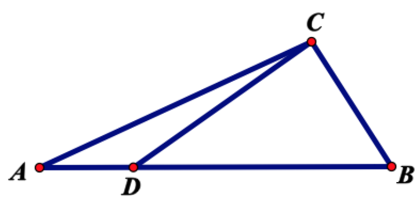
Remark: Googling this will yield more proofs of the pythagorean theorem than you can absorb in a lifetime. Many of them are simple scissors-congruence constructions.

8 *Medians*. We all know that the three medians of a triangle (the lines going from a vertex to the midpoint of the opposite side) intersect in a point and that the intersection point cuts the medians in a $1:2$ ratio. Accept, for the moment, that the medians meet in a point (although that is worth proving from scratch), but use your knowledge about area to deduce the $1:2$ ratio, with ease!

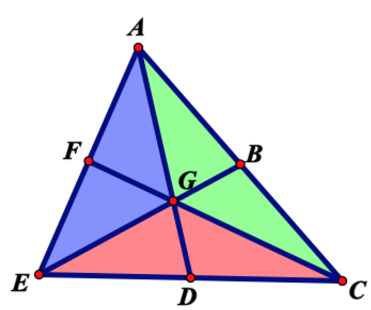
Solution: First, we recall the principle—an easy consequence of the formula for the area of a triangle—that if two triangles share the same apex and same base line, then the ratio of the areas is just the ratio of the base lengths. In the picture below, triangles ADC and DBC share vertex C and have the same base line AB. Thus they have the same height (the perpendicular from C dropped to the base line), and thus

$$\frac{[ADC]}{[DBC]} = \frac{AD}{DB},$$

where we use the notation $[ADC]$ to indicate the area of a polygon.

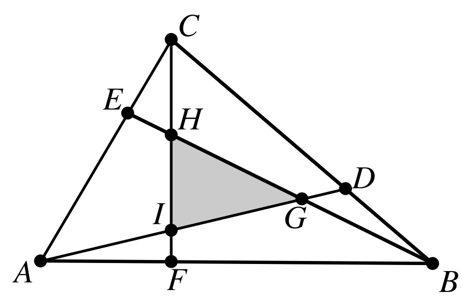


Now, let's apply this idea to medians. In the picture below, B, D, F are respectively the midpoints of sides AC, CE, EA. Using the ratio principle, we immediately see that the two pink triangles (EDG, CDG) have equal area. Likewise, the two green triangle and the two blue triangles have equal area. But it is *also* true that $[AED] = [ACD]$; i.e. the area of one pink plus two blue equals one pink plus two green. But this means that two blue equals two green, so green equals blue! Likewise (using a different apex), we can see that blue equals pink.

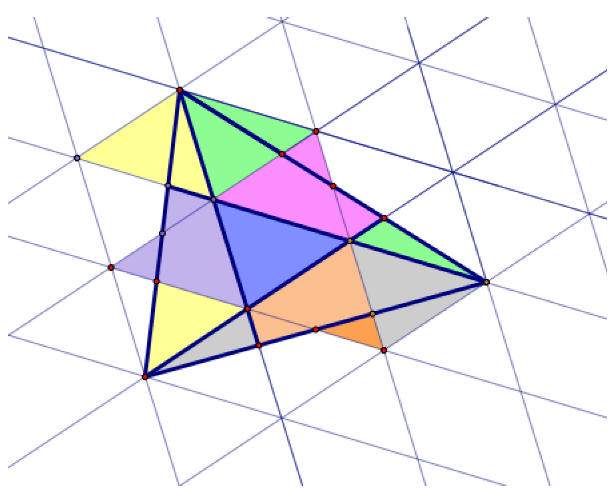


In other words, the six smaller triangles all have equal area. Finally, look at triangles AFG and ACG. We know that the second one has twice the area of the first, and they share the apex A and base line FC. So the ratio of $FG : GC = 1 : 2$. Of course the same argument works on the other medians.

9 *Threedians*. The lines AD, BE, CF below are "threedians;" in other words, they hit the opposite edges at a trisection point ($CE = AC/3$, etc.). What can you say about the relationship of the shaded area $[GHI]$ to the area $[ABC]$?



Solution: Using the same translation idea as in problem 2, we can deduce that the shaded area (now colored blue) is always one-seventh of the total area of the triangle. Ponder this picture until it becomes obvious! It's really beautiful!



10 *Infinite dissections*. (Thanks to Sam Vandervelde.) Can you dissect a square into infinitely many line segments? Of course you can. (A line segment, by the way, is straight and has two endpoints and infinitely many points in-between; in other words, it has positive, non-zero length. A single point is *not* a line segment. And "dissecting into line segments" means decomposing into *disjoint* line segments (every point of the target shape is covered by a line segment, and no two line segments have *any* point in common, and no other points are covered). So using this definition of line segment, here are a few harder questions. Which of the following can you dissect into infinitely many line segments?

(a) A rectangle.

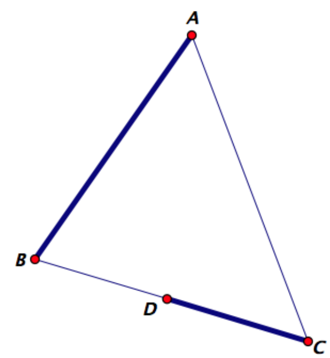
Solution: This is pretty obvious. Just draw parallel line segments.

(b) A trapezoid.

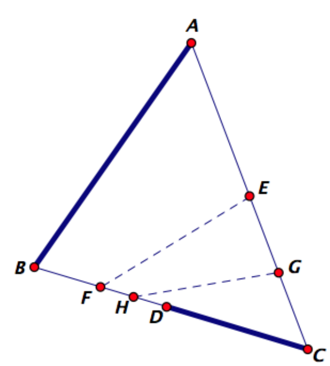
Solution: So is this.

(c) A triangle.

Solution: This is a nice example of using wishful thinking. If only a triangle were a trapezoid, say. *Make it so!*

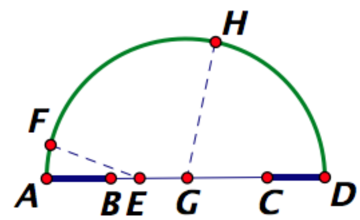


Who said that any of the sides of the trapezoid need to be parallel? Now we proceed as if we had a trapezoid. The two shaded lines AB and CD are the initial two line segments of the dissection and then we can draw line segments that include the rest of the triangle as follows: Along line BD, for example, the midpoint is F. We join F with the midpoint of AC. Likewise, point H is $3/4$ of the way from B to D, so we join it to G, which is also $3/4$ of the way between A and C. In this way, we can pick any point on BD and find a corresponding point on AC to join it to. The picture below illustrates this. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin lines are not one of the dissection line segments, but merely guidelines.)



(d) A semicircle.

Solution: We use the same idea as with the triangle, only twice. Start with segments AB and CD. Then we imagine that these two segments are the sides of a "trapezoid" whose top is the arc of the semicircle and whose bottom is BC. For example, EF and GH are two of the segments in the dissection. (Note that in this picture, the line segments are indicated by thick lines or dashed lines; the thin line is not one of the dissection line segments, but merely a guideline.)



If you are not satisfied with this picture, and want a formula, you can assign to each point on the interval between B and C an angle between 180 and 0 degrees, and then join each point on the BC to the corresponding point on the arc with that angle. For example, B joins with A (angle 180), and C joins with D (angle is 0) and E joins with 170 degrees, G joins with 77 degrees, etc.

(e) A circle.

Solution: Here's a picture to illustrate the idea. Essentially, we take two "semicircles" and put them together, but leave space for a "belt" of parallel lines. Again, the thick and dashed lines are line segments used in the dissection, but the thin lines are merely guidelines.