Investigation 101 — Monday AM

MitM TIP 2023 — Paul Zeitz, zeitzp@usfca.edu

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1 *Know your sequences!*. You already know about odds and evens, but you need to have, at the very least, passive familiarity with as many other sequences as possible. Here are a few.

- The **triangular numbers** are the sums of consecutive integers, starting with 1. The first few are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, \ldots$.
- The **squares** are $1, 4, 9, 16, 25, 26, 49, 64, 81, 100, 121, 144, 169, 196, 225 \ldots$.
- The **powers of two** are the numbers of the form 2^k for non-negative integers k. The first few terms are $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots$, since $2^0 = 1$.
- The **Fibonacci numbers** f_n are defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

2 *Investigating these sequences*. Try to ask questions that involve one or more sequence, and then investigate them. Here are a few suggestions.

- Is there a relationship between odd numbers and squares?
- Are square numbers ever triangular numbers as well?
- Make a list of triangular numbers; then look at what happens when you multiply each number by 8 and then add 1.

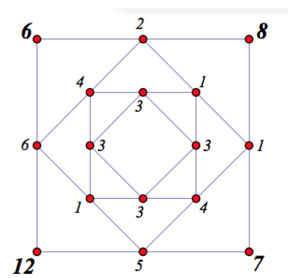
3 *Adding and multiplying*. Here's a fun and simple activity: Write a multiplication table, say from 1×1 up to 10×10, and figure out how to add up the entries of the table. Without doing a lot of work!

4 A number is called **trapezoidal** if it can be expressed as a sum of two or more consecutive positive integers. For example, $7 = 3 + 4$ and $10 = 1 + 2 + 3 + 4$ and $12 = 3 + 4 + 5$ are all trapezoidal. Investigate, generate questions, come up with conjectures.

5 *Fibonacci investigations*. Here are just a few suggestions.

- Investigate parity (odd or even), divisibility by 3, divisibility by 5, perfect squares, etc. for the Fibonacci numbers.
- Try adding the Fibonacci numbers.
- Try adding squares of Fibonacci numbers.

6 *The difference game*. Start by labeling the vertices of a square with numbers. Then write the difference of the values at two adjacent vertices on the midpoint of the line joining them; this produces four new values at the vertices of a smaller square. Keep repeating this process, generating smaller and smaller squares until the process ends. In the example below, we started with the values $6, 8, 7, 12$ (shown in larger font) which generated the values $2, 1, 5, 6$, then $4, 1, 4, 1$. The final square shown has all vertices equal to 3; clearly the next square (and all subsequent squares) will have only zeros at each vertex.



Investigate, generate questions, come up with conjectures.

7 *Pizza slicing*. Imagine a giant pizza. For each n, what is the maximum number of pieces you can get if you slice this pizza with straight line cuts? The lines are *infinite*; they are not line segments.

That's the warm up. Now investigate a slight modification that makes the problem even more interesting: Remove the words "the maximum" above, and replace it with "are the possible." For example, if $n = 3$, the *maximum* number of pieces will be 7 (verify!) but it is possible to get fewer. If all three lines coincide, you will get just 2 pieces. If all three lines are parallel, you will get 4 pieces. If two are parallel, and one is not, you get 6.