MitM TIP 2023 — Monday

Numbers and Investigation

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1 *Know your sequences!*. You already know about odds and evens, but you need to have, at the very least, passive familiarity with as many other sequences as possible. Here are a few.

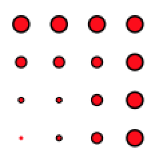
- The **triangular numbers** are the sums of consecutive integers, starting with 1. The first few are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, \ldots$.
- The **squares** are $1, 4, 9, 16, 25, 26, 49, 64, 81, 100, 121, 144, 169, 196, 225 \ldots$.
- The **powers of two** are the numbers of the form 2^k for non-negative integers k. The first few terms are $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots$, since $2^0 = 1$.
- The **Fibonacci numbers** f_n are defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

2 *Investigating these sequences*. Try to ask questions that involve one or more sequence, and then investigate them. Here are a few suggestions.

- Is there a relationship between odd numbers and squares?

  Solution: Indeed, this picture shows that the nth square is the sum of the first n odd numbers, explicitly demonstrating that $1 + 3 + 5 + 7 = 4^2$.

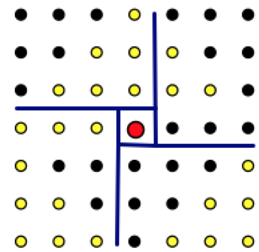
  

- Are square numbers ever triangular numbers as well?

  Solution: We didn't really look into this, other than observe that $t_1 = 1$ and $t_8 = 36$. A computer or calculator will allow you to discover far more triangular numbers that are also squares; it is a fascinating exploration that leads to interesting patterns that amazingly, involve $\sqrt{2}$.

- Make a list of triangular numbers; then look at what happens when you multiply each number by 8 and then add 1.

  Solution: Proof by picture shows that if $T = 1 + 2 + 3$, then 8 copies of T can be arranged in a "pinwheel" pattern that makes a perfect square minus 1 (the red dot).



3 *Adding and multiplying*. Here's a fun and simple activity: Write a multiplication table, say from 1×1 up to 10×10, and figure out how to add up the entries of the table. Without doing a lot of work!

Solution: A little experimenting with smaller tables gets us the nice answer of t_n^2 for an $n \times n$ table. To see why, look at the multiplication table tor $n = 3$.

	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

The sum of the entries is

$$(1+2+3)+2(1+2+3)+3(1+2+3) = (1+2+3)(1+2+3).$$

4 A number is called **trapezoidal** if it can be expressed as a sum of two or more consecutive positive integers. For example, $7 = 3+4$ and $10 = 1+2+3+4$ and $12 = 3+4+5$ are all trapezoidal. Investigate, generate questions, come up with conjectures.

Solution: The main thing that we determined is that a number is trapezoidal if and only if it is NOT a power of 2. In other words, if it does not have an odd factor. Thus we need to show two things:

1. If a number is trapezoidal, it must have an odd factor (besides 1, of course).
2. If a number has an odd factor (besides 1) it is trapezoidal.

To prove #1, we observe that if a number is trapezoidal with an odd number of addends, then it automatically has an odd factor. For example, the 5-addend sum $10+11+12+13+14+15 = 5 \cdot 12$. This is because of the "balance-beam" principle, that the numbers all balance around the center point of 12.

But what if there are an even number of addends? Then we use the fact that the first and last numbers are of different parity, so their sum is odd, and we can collect several copies of this odd number. For example, consider the 8-addend sum

$$16+17+18+19+20+21+22+23.$$

There is no central balance point, but instead we just observe that the entire sum is equal to

$$(16+23)+(17+22)+(18+21)+(19+20)=4\cdot 39,$$

a multiple of the odd number 39. Clearly this method works for any even-addend trapezoidal number.

Thus, we know that trapezoidal numbers have odd factors, so they cannot be powers of two. But we haven't shown that ALL numbers with odd factors can be written "trapezoidally," which is what #2 asserts So let's prove that.

Suppose we have a number which has an odd factor, say $21=3\cdot 7$. We can write

$$21=7+7+7=6+7+8,$$

where we used the "balance-beam" principle.

But this method doesn't always work smoothly. Suppose we have $22=2\cdot 11$. We write

$$22=2+2+2+2+2+2+2+2+2+2+2,$$

and the balance-beam method yields

$$22=(-3)+(-2)+(-1)+0+1+2+3+4+5+6+7,$$

which is a sum of 11 consecutive integers, but of course they are not all positive. But so what! Just let the negatives cancel out their positive sisters, and we get the sum

$$22=4+5+6+7,$$

so it is trapezoidal.

5 *Fibonacci investigations*. Here are just a few suggestions.

- Investigate parity (odd or even), divisibility by 3, divisibility by 5, perfect squares, etc. for the Fibonacci numbers.
- Try adding the Fibonacci numbers.
- Try adding squares of Fibonacci numbers.

6 *Added during the session*. Given an $n\times n$ grid, count the number of squares possible. Call this number S_n. Also count the number of rectangles possible. Call this R_n. For example, you should be able to verify that $S_1=R_1=1$, and $S_2=5, R_2=9$, and $S_3=14, R_3=36$. Can you find formulas for S_n and R_n?

Solution: Let's try to compute R_3 by drawing a 3×3 grid, where we label each grid square with the number of rectangles possible that have that grid square as its upper-left corner.

9	6	3
6	4	2
3	2	1

Notice that for each square we label, we are merely counting the number of squares that there are to the right and below our square (including our square itself), since any of the other squares to the right and below can serve as the lower-right corner of the rectangle. We are exactly doing the multiplication table problem (problem 3), so the answer is t_3, and in general we have deduced that

$$R_n = t_n^2.$$

Let's now do the same analysis for S_3. We get the following grid.

3	2	1
2	2	1
1	1	1

for a total of 14. Likewise, the grid for S_4 is

4	3	2	1
3	3	2	1
2	2	2	1
1	1	1	1

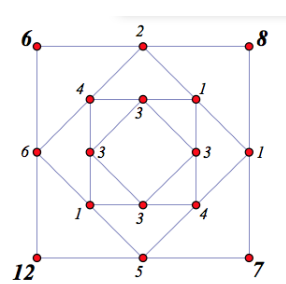
Let's think recursively. How do we get from S_3 to S_4? We merely increase every value of the S_3 grid by 1, and then add a boundary layer of 1s on the right and bottom. Since we are interested in the *sum* of the grid values, we are just adding a 4×4 grid consisting of all 1s to the S_3 grid sum. In other words,

$$S_4 = S_3 + 4^2.$$

Since $S_1 = 1 = 1^2$, we have deduced that in general,

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

7 *The difference game*. Start by labeling the vertices of a square with numbers. Then write the difference of the values at two adjacent vertices on the midpoint of the line joining them; this produces four new values at the vertices of a smaller square. Keep repeating this process, generating smaller and smaller squares until the process ends. In the example below, we started with the values $6, 8, 7, 12$ (shown in larger font) which generated the values $2, 1, 5, 6$, then $4, 1, 4, 1$. The final square shown has all vertices equal to 3; clearly the next square (and all subsequent squares) will have only zeros at each vertex.



Investigate, generate questions, come up with conjectures.

Solution: There are two big questions (at least). The first is whether you always end up with zeros eventually, and the second is whether you can hold out for an arbitrary length of time.

The answers are YES to both questions (provided you start with positive integers). To see why you eventually get only zeros, we will verify two simple observations.

- **The maximum of the four values NEVER goes up; it either stays the same or decreases.** This is pretty obvious. However, this is not enough to force the values to all become zero, since perhaps there could be some "oscillation." We need the next observation.
- **Eventually all the values will be even.** This is not obvious, but there are only 6 different parity cases to try, using 0 for even and 1 for odd: 0000,1000,1100,1010,1110,1111. With each case, we end up with all evens (all 0s) in at most 4 turns.

Thus, no matter what numbers you get, eventually, you will end up with a square whose values are all even. Call the values $2a, 2b, 2c, 2d$. Now, when you continue the process, everything is multiplied by this factor of 2, and you can visualize it as two identical squares playing the difference game. At some point, both of *those* squares will have all even values. But it was really just one square, so putting the factor of 2 back, we now have all values being *multiples of 4*.

This process will continue indefinitely, which forces the values to eventually be zeros, since no positive integer can have an arbitrarily high power of two dividing it!

Now that we know we will eventually get all zeros, we need to find a way to hold off that fate as long as possible. We can design a starting square that will take at least N turns to zero out, for any N. The secret is "tribonacci numbers," the sequence $1, 1, 1, 3, 5, 9, 17, \ldots$, where the first three terms are 1 and each subsequent term is the sum of the *three* preceding terms.

To see how this works, imagine that we put the values $t_{13}, t_{12}, t_{11}, t_{10}$ on the vertices of a square (clockwise, in that order), where t_n is the $n$the tribonacci number. After one turn, the vertices of the new square are

$$t_{13} - t_{12}, \quad t_{12} - t_{11}, \quad t_{11} - t_{10}, \quad t_{13} - t_{10}.$$

Using the definition of tribonacci numbers, we see that

$$t_{13} - t_{12} = (t_{12} + t_{11} + t_{10}) - t_{12} = t_{11} + t_{10}.$$

Likewise, $t_{12} - t_{11} = t_{10} + t_9$ and $t_{11} - t_{10} = t_9 + t_8$. The last difference is a slightly different pattern (since the indices differ by 3 instead of 1), but the magic of tribonacci yields

$$t_{13} - t_{10} = (t_{12} + t_{11} + t_{10}) - t_{10} = t_{12} + t_{11}.$$

It doesn't matter which vertex is "first," as long as we go in order. So observe that the values of this new square, going clockwise, are

$$t_{12} + t_{11}, \quad t_{11} + t_{10}, \quad t_{10} + t_9, \quad t_9 + t_8.$$

Now if we use the ”two squares” idea, we see that this new square can really be thought of as one square with vertices of $t_{12}, t_{11}, t_{10}, t_9$, and the other, with corresponding vertices of t_{11}, t_{10}, t_9, t_8. Well, we know what happens when we take differences; these are just squares with consecutive tribonacci values (only shifted backwards a bit).

You can see the pattern. If we let S_n denote the square whose values are the consecutive tribonacci numbers $t_n, t_{n-1}, t_{n-2}, t_{n-3}$, we see that after one turn, S_n becomes ”$S_{n-1} + S_{n-2}$,” and after k turns, we will have a ”sum” of 2^k tribonacci squares. The values of these tribonacci squares stay non-zero and non-constant until you hit S_4; after four turns that zeros out.

Notice that this ”multiple squares” idea requires that the values of the squares be monotonic in the same direction; this allows us to couple or uncouple squares without interference. So at the very least, we can be assured of getting at least $n-4$ turns if we start with S_n.

8 *Pizza slicing*. Imagine a giant pizza. For each n, what is the maximum number of pieces you can get if you slice this pizza with straight line cuts? The lines are *infinite*; they are not line segments.

That’s the warm up. Now investigate a slight modification that makes the problem even more interesting: Remove the words “the maximum” above, and replace it with “are the possible.” For example, if $n = 3$, the *maximum* number of pieces will be 7 (verify!) but it is possible to get fewer. If all three lines coincide, you will get just 2 pieces. If all three lines are parallel, you will get 4 pieces. If two are parallel, and one is not, you get 6.

Solution: The answer to the warm up is a classic problem; for n lines it is just 1 more than the nth triangular number. I have no idea about the more general problem. It is fun and rich and there are plenty of partial solutions possible.