20th Bay Area Mathematical Olympiad

BAMO 2018 Problems and Solutions

February 27, 2018

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

A Twenty-five people of different heights stand in a 5×5 grid of squares, with one person in each square. We know that each row has a shortest person; suppose Ana is the tallest of these five people. Similarly, we know that each column has a tallest person; suppose Bev is the shortest of these five people.

Assuming Ana and Bev are not the same person, who is taller: Ana or Bev? Prove that your answer is always correct.

Solution. Bev is taller. We consider three possible cases.

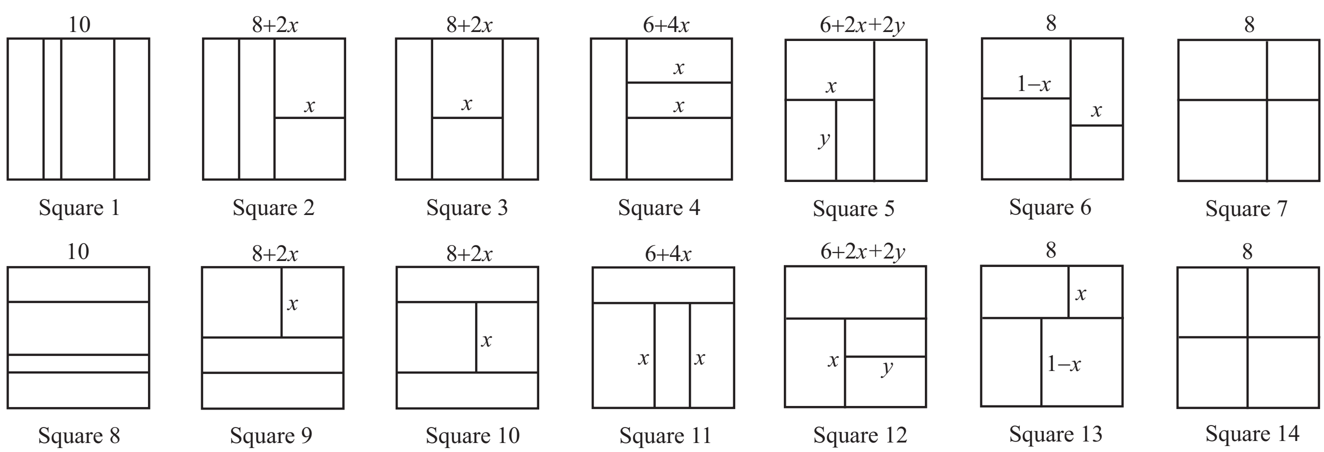
Case 1: If Ana and Bev are in the same row, then Bev is taller because Ana is by definition the shortest in that row.

Case 2: If Ana and Bev are in the same column, then Bev is taller because Bev is by definition the tallest in that column.

Case 3: If Ana and Bev are in neither the same row nor the same column, then let "Casey" be the person at the intersection of Ana's row and Bev's column. Then Ana is shorter than Casey, and Casey is shorter than Bev, so Ana is shorter than Bev.

B A square with sides of length 1 cm is given. There are many different ways to cut the square into four rectangles. Let S be the sum of the four rectangles' perimeters. Describe all possible values of S with justification.

Solution. The answer is $6 < S \leq 10$. This can be shown by considering several cases, as shown in the 14 figures below.



Observe that in every case, there is either a horizontal or a vertical line segment of length 1 drawn inside the square, as well as some additional segments. For example, there are three such *vertical* lines inside Square 1, two inside each of Squares 2-3, and one inside each of Squares 4-7. The figures in the first row are rotated by 90° to give corresponding partitions of the square in the second row, each with a *horizontal* line cutting through the whole square; except for Square 14, which displays two lines, a horizontal and a vertical one, that pass through the center of the square.

The total perimeter S of the four rectangles in each case is the original perimeter 4 of the square, plus twice the length of all line segments drawn inside the square since each of these must be counted twice for the two (or more) rectangles that include them in their perimeters. So we must have $S = 4+2\cdot 1+2a = 6+2a$, where $2\cdot 1$ stands for twice a segment that crosses all the way through the square and a stands for any additional segment(s) inside the square. Since $a > 0$ in all cases, we have $S > 6$.

The maximum of $S = 10$ is achieved by cutting the square into four $1\times\frac{1}{4}$ rectangles with three parallel cuts, as in Squares 1 and 8.

All values in between can be achieved as well. Indeed, the total perimeters S are written on top of each of the 14 cases. By shifting left or right the length 1 vertical segments in Squares 2,3,4, or by shifting up or down the length 1 horizontal segments in Squares 9,10,11, we can make the length x vary from 0 to 1: $0 < x < 1$. Thus, for example, Square 4 can achieve any total perimeter $S = 6+4x$ between 6 and 10: $6 < S < 10$. In a similar way, Squares 5 and 12 can be drawn with lengths x and y varying between 0 and 1, and so these squares can also achieve any perimeter $S = 6+2x+2y$ between 6 and 10.

Putting everything together, S can be any number in the interval $(6,10]$; i.e. $6 < S \leq 10$. $\square$

C/1 An integer c is *square-friendly* if it has the following property: For every integer m, the number $m^2+18m+c$ is a perfect square. (A perfect square is a number of the form n^2, where n is an integer. For example, $49 = 7^2$ is a perfect square while 46 is not a perfect square. Further, as an example, 6 is not square-friendly because for $m = 2$, we have $(2)^2+(18)(2)+6 = 46$, and 46 is not a perfect square.)

In fact, exactly one square-friendly integer exists. Show that this is the case by doing the following:

(a) Find a square-friendly integer, and prove that it is square-friendly.

(b) Prove that there cannot be two different square-friendly integers.

Solution.

(a) $c = 81$ is square-friendly, since $m^2+18m+81 = (m+9)^2$ which is an integer whenever m is an integer.

(b) Suppose there are two different square-friendly integers c and c'. In that case, for every integer m, we have that $m^2+18m+c$ and $m^2+18m+c'$ are perfect squares. Thus we have infinitely many pairs of perfect squares that differ by $c-c'$. However, this is not possible, because the differences between consecutive squares are $1,3,5,7,9,11,\ldots$. Eventually these differences are greater than $c-c'$, so no perfect square beyond that point differs from another perfect square by $c-c'$. Thus we have a contradiction, and so there cannot be two different square-friendly integers.

D/2 Let points P_1, P_2, P_3, and P_4 be arranged around a circle in that order. (One possible example is drawn in Diagram 1.) Next draw a line through P_4 parallel to $\overline{P_1P_2}$, intersecting the circle again at P_5. (If the line happens to be tangent to the circle, we simply take $P_5 = P_4$, as in Diagram 2. In other words, we consider the second intersection to be the point of tangency again.) Repeat this process twice more, drawing a line through P_5 parallel to $\overline{P_2P_3}$, intersecting the circle again at P_6, and finally drawing a line through P_6 parallel to $\overline{P_3P_4}$, intersecting the circle again at P_7. Prove that P_7 is the same point as P_1.

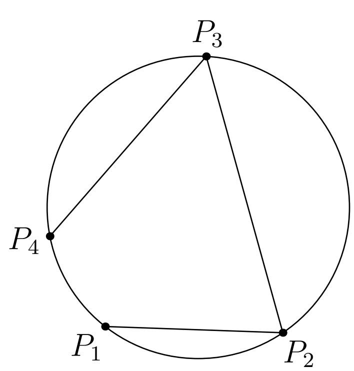


Diagram 1

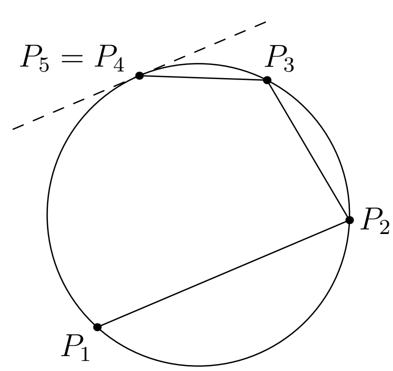
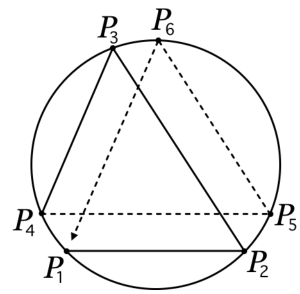


Diagram 2

Solution. We claim that arcs $\widehat{P_1P_4}$ and $\widehat{P_2P_5}$ are congruent. This follows at once from the fact that these arcs are intercepted by angles $\angle P_1P_5P_4$ and $\angle P_5P_1P_2$, which are congruent because they are alternate interior angles to parallel segments P_1P_2 and P_4P_5. Furthermore, note that these two arcs are oppositely oriented; thus in the diagram P_4 is situated in a clockwise direction away from P_1 while P_5 is counterclockwise away from P_2. p In the same manner, arcs $\widehat{P_3P_6}$ and $\widehat{P_4P_7}$ are also congruent to $\widehat{P_1P_4}$, oriented in the same and opposite ways to $\widehat{P_1P_4}$, respectively. In summary, arcs $\widehat{P_1P_4}$ and $\widehat{P_4P_7}$ are congruent and oppositely oriented, which implies that points P_1 and P_7 coincide.



Bonus Solution. *This student-written solution won the BAMO Brilliance Award.*

We use complex numbers. Let $|a|$, $\overline{a}$ denote the magnitude and complex conjugate of a, respectively.

Without loss of generality, P_1, P_2, P_3, P_4 are on the unit circle. Let $P_1 = a, P_2 = b, \ldots, P_7 = g$ as complex numbers. Note that $|a| = |b| = |c| = |d| = |e| = |f| = |g| = 1 \implies \overline{a} = \frac{1}{a}, \overline{b} = \frac{1}{b}, \overline{c} = \frac{1}{c}, \overline{d} = \frac{1}{d}, \overline{e} = \frac{1}{e}, \overline{f} = \frac{1}{f}, \overline{g} = \frac{1}{g}$. Then

$$\begin{aligned}
P_4P_5 \parallel P_1P_2 &\implies \frac{d-e}{a-b} = \overline{\frac{d-e}{a-b}} \\
&\implies \frac{d-e}{a-b} = \frac{\frac{1}{d}-\frac{1}{e}}{\frac{1}{a}-\frac{1}{b}} \\
&\implies \frac{d-e}{a-b} = \left(\frac{e-d}{b-a}\right)\left(\frac{ab}{de}\right) \\
&\implies \frac{ab}{de} = 1 \\
&\implies e = \frac{ab}{d}.
\end{aligned}$$

In an analogous way, $P_5P_6 \parallel P_2P_3 \implies f = \frac{bc}{e}$ and $P_6P_7 \parallel P_3P_4 \implies g = \frac{cd}{f}$.

Then

$$P_7 = g = \frac{cd}{f} = \frac{cd}{\left(\frac{bc}{e}\right)} = \frac{cde}{bc} = \frac{de}{b} = \frac{d\left(\frac{ab}{d}\right)}{b} = \frac{ab}{b} = a = P_1,$$

and we are done.

E/3 Suppose that 2002 numbers, each equal to 1 or -1, are written around a circle. For every two adjacent numbers, their product is taken; it turns out that the sum of all 2002 such products is negative. Prove that the sum of the original numbers has absolute value less than or equal to 1000. (The absolute value of x is usually denoted by $|x|$. It is equal to x if $x \geq 0$, and to $-x$ if $x < 0$. For example, $|6| = 6$, $|0| = 0$, and $|-7| = 7$.)

Solution. Suppose that n of the original numbers are $+1$, so that the remaining $2002 - n$ are -1, and their sum is

$$1 \cdot n + (-1) \cdot (2002 - n) = 2n - 2002.$$

Also suppose that m of the products equal -1; then the remaining $2002 - m$ equal 1, and the sum of the products is

$$1 \cdot (2002 - m) + (-1) \cdot m = 2002 - 2m.$$

Since this is negative, $m > 1001$. Now, each product of -1 must come from a $+1$ and a -1 among the original numbers, and each -1 among the original numbers can contribute to at most two such products; hence the original numbers include more than $1001/2 > 500$ -1's. Thus,

$$2002 - n \geq 501 \Rightarrow n \leq 1501 \Rightarrow 2n - 2002 \leq 1000.$$

Similarly, the original numbers include more than 500 $+1$'s, so

$$n \geq 501 \Rightarrow 2n - 2002 \geq -1000.$$

Thus, $-1000 \leq 2n - 2002 \leq 1000$, which is what we need.

4 (a) Find two quadruples of positive integers (a,b,c,n), each with a different value of n greater than 3, such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = n$.

(b) Show that if a,b,c are nonzero integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer, then abc is a perfect cube. (A perfect cube is a number of the form n^3, where n is an integer.)

Solution.

(a) For example, $(1,2,4,5)$ and $(9,2,12,6)$ work.

(b) Before solving the problem, we establish a useful definition and lemma.

If p is a prime and x is a nonzero rational number, we define $\operatorname{ord}_p x$ to be the unique integer k such that $p^{-k}x$ is an integer not divisible by p. For example, $\operatorname{ord}_3 45 = 2$ and $\operatorname{ord}_5 \frac{11}{5} = -1$. Then we claim the following:

Lemma. Let p be prime and let x,y be nonzero rational numbers. Then:

(i) $\operatorname{ord}_p(xy) = \operatorname{ord}_p x + \operatorname{ord}_p y$.

(ii) $\operatorname{ord}_p(x/y) = \operatorname{ord}_p x - \operatorname{ord}_p y$.

(iii) If $\operatorname{ord}_p x < \operatorname{ord}_p y$, then $\operatorname{ord}_p(x+y) = \operatorname{ord}_p x$.

Proof of the lemma. Parts (i)–(ii) are straightforward. For part (iii), let $k = \operatorname{ord}_p x$ and $k + \ell = \operatorname{ord}_p y$, where we assume $\ell > 0$. Then $p^{-k}x = n$ and $p^{-k-\ell}y = m$ for some integers m,n not divisible by p. It follows that $p^{-k}(x+y) = n + p^{\ell}m$, and this is an integer not divisible by p. Thus $\operatorname{ord}_p(x+y) = k$ as claimed, proving the lemma. $\square$

Now we turn to the problem.

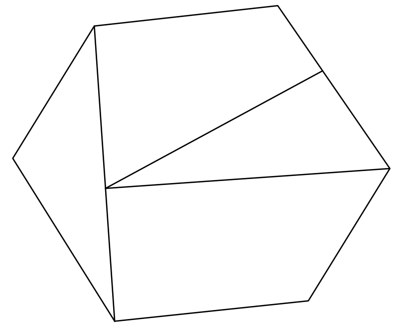
Suppose a,b,c are integers such that $a/b + b/c + c/a$ is an integer. We will show that $\operatorname{ord}_p(abc)$ is a multiple of 3 for all primes p, which implies that abc is a perfect cube.

Let p be a prime and let $r = \operatorname{ord}_p a$, $s = \operatorname{ord}_p b$, $t = \operatorname{ord}_p c$. By part (i) of the lemma, $\operatorname{ord}_p(abc) = r + s + t$.

By part (ii) of the lemma, we have $\operatorname{ord}_p(a/b) = r - s$, $\operatorname{ord}_p(b/c) = s - t$, and $\operatorname{ord}_p(c/a) = t - r$. If these three differences are 0, then $r = s = t$ and $r + s + t$ is trivially a multiple of 3. Otherwise, the least of $r - s$, $s - t$, $t - r$ is negative and the greatest is positive (since their sum is 0).

We now claim that at least two of $r - s$, $s - t$, $t - r$ must be tied for least. If this is not true, then, by applying part (iii) of the lemma, we may conclude that $\operatorname{ord}_p(a/b + b/c + c/a) = \min\{r - s, s - t, t - r\} < 0$, which contradicts the assumption that $a/b + b/c + c/a$ is an integer. Thus we have proven our claim. But if two of $r - s$, $s - t$, $t - r$ are equal, then r,s,t (in some order) form an arithmetic progression, and $r + s + t$ is three times the middle term of the progression. Thus we have shown that $r + s + t$ is a multiple of 3; in other words, $\operatorname{ord}_p(abc)$ is a multiple of 3 for all primes p, and we are finished.

5 To *dissect* a polygon means to divide it into several regions by cutting along finitely many line segments. For example, the diagram below shows a dissection of a hexagon into two triangles and two quadrilaterals:

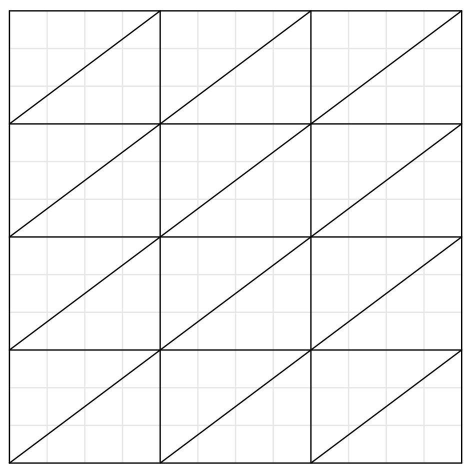


An *integer-ratio right triangle* is a right triangle whose side lengths are in an integer ratio. For example, a triangle with sides $3,4,5$ is an integer-ratio right triangle, and so is a triangle with sides $\frac{5}{2}\sqrt{3}, 6\sqrt{3}, \frac{13}{2}\sqrt{3}$. On the other hand, the right triangle with sides $\sqrt{2}, \sqrt{5}, \sqrt{7}$ is not an integer-ratio right triangle.

Determine, with proof, all integers n for which it is possible to completely dissect a regular n-sided polygon into integer-ratio right triangles.

Solution. Abbreviate integer-ratio right triangle by IRRT.

The square ($n = 4$) has such a decomposition. For example, a 12×12 square can be cut into twenty-four 3–4–5 triangles as shown below:



Now we show that $n = 4$ is the only solution. The proof is by contradiction. Suppose an n-gon with $n \neq 4$ (and $n \geq 3$) has a dissection into IRRTs. Choose an arbitrary vertex P of the n-gon. One or more IRRTs meet at P; let their internal angles at P be $\theta_1, \theta_2, \ldots, \theta_k$, where we have

$$\theta_1 + \theta_2 + \cdots + \theta_k = \frac{n-2}{n} \cdot \pi. \tag{1}$$

Observe that $\sin\theta_i$ and $\cos\theta_i$ are rational for each $i = 1, \ldots, k$. By applying angle-sum identities, we may infer that the sine and cosine of $\frac{n-2}{n} \cdot \pi$, and thus of $2\pi/n$, are rational as well. Applying angle-sum identities again, it follows that for any divisor m of n, the sine and cosine of $2\pi/m$ are also rational. Since $n \neq 4$ and $n \geq 3$, n has a divisor which is either 8 or an odd prime. So, we can restrict our attention to these two cases.

For $n = 8$, we have $\sin(2\pi/8) = \sqrt{2}/2$. This is irrational, so we have our contradiction.

Now we consider the case $n = p$ where p is an odd prime. Let $t = \tan(2\pi/p)$, which must be rational. By the multiple-angle identity for tangents, t satisfies

$$pt - \binom{p}{3}t^3 + \binom{p}{5}t^5 - \cdots \pm t^p = 0. \tag{2}$$

By the Rational Root Theorem, the only possible rational values of t are $t = \pm 1$ and $t = \pm p$. But $t \neq \pm 1$, because $\frac{2\pi}{p} \neq \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. If $t = \pm p$, then the left-hand side of (2) is congruent to $p^2 \pmod{p^3}$ and thus cannot equal 0. Again, we have a contradiction.

It follows that no n-gon with $n \neq 4$ has a dissection into IRRTs.