2nd Bay Area Mathematical Olympiad

February 29, 2000

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. We don't expect many students to solve all the problems; indeed, solving just one problem completely is a fine achievement. We do hope, however, that you find the experience of thinking deeply about mathematics for 4 hours to be a fun and rewarding challenge. We hope that you find BAMO interesting, and that you continue to think about the problems after the exam is over. Good luck!

Problems

1 Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1. (Two integers are relatively prime if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus $37 = 22+15$ represents the number 37 in the desired way.)

2 Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F; let k_2 be the circle passing through points B, E, and D; and let k_3 be the circle passing through C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

3 Let $x_1, x_2, \ldots, x_n$ be positive numbers, with $n \geq 2$. Prove that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\cdots\left(x_n + \frac{1}{x_n}\right) \geq \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right).$$

4 Prove that there exists a set S of 3^{1000} points in the plane such that for each point P in S, there are at least 2000 points in S whose distance to P is exactly 1 inch.

Please turn over for problem #5.

5 Alice plays the following game of solitaire on a 20×20 chessboard. She begins by placing 100 pennies, 100 nickels, 100 dimes, and 100 quarters on the board so that each of the 400 squares contains exactly one coin. She then chooses 59 of these coins and removes them from the board. After that, she removes coins, one at a time, subject to the following rules:

- A penny may be removed only if there are four squares of the board adjacent to its square (up, down, left, and right) that are vacant (do not contain coins). Squares "off the board" do not count towards this four: for example, a non-corner square bordering the edge of the board has three adjacent squares, so a penny in such a square cannot be removed under this rule, even if all three adjacent squares are vacant.
- A nickel may be removed only if there are at least three vacant squares adjacent to its square. (And again, "off the board" squares do not count.)
- A dime may be removed only if there are at least two vacant squares adjacent to its square ("off the board" squares do not count).
- A quarter may be removed only if there is at least one vacant square adjacent to its square ("off the board" squares do not count).

Alice wins if she eventually succeeds in removing all the coins. Prove that it is impossible for her to win.

You may keep this exam. Please remember your ID number! Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2000 Awards Ceremony**, which will be held at the University of San Francisco from 11–2 on Sunday, March 12. This event will include lunch (free of charge), a mathematical talk by Professor Persi Diaconis of Stanford University, and the awarding of over 60 prizes, worth approximately $5000 in total. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.

 For more information about the awards ceremony or other questions about BAMO, please contact Paul Zeitz (415-422-6590, zeitz@usfca.edu), Zvezdelina Stankova-Frenkel (510-430-2144, stankova@mills.edu), or Joe Buhler (510-643-6056, jpb@msri.org).

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Solutions

1 We shall present four solutions. The first solution, the most elementary, is similar to most of the correct solutions that were submitted. The second solution uses knowledge about the Euler ϕ-function, which counts the number of integers which are relatively prime to a given number. The final two solutions use insights about the distribution of prime numbers.

Solution 1: First note that if d divides two integers a and b, then d must also divide their difference $a-b$. Therefore, consecutive positive integers are always relatively prime (difference is 1). Likewise, if a and b are both odd with a difference of 2 or 4, then a and b are relatively prime.

Now let n be greater than 7.

- If n is odd, then n has the form $n = 2k + 1$, where $k \geq 3$ is an integer. Thus we write the sum $n = k + (k + 1)$.
- If n is even, then n has the form $n = 2k$, where $k \geq 4$ is an integer.
  - If k is even, we write $n = (k - 1) + (k + 1)$.
  - If k is odd, we write $n = (k - 2) + (k + 2)$. ■

Solution 2 (sketch): Recall that (x, y) denotes the greatest common divisor of the natural numbers x and y. To write n as $a + b$ with $(a, b) = 1$ and $a, b > 1$ is the same as to find a satisfying $(a, n - a) = 1$ and $1 < a < n - 1$. But $(a, n - a) = (a, n)$, so for $n > 2$, the number of such a is $\phi(n) - 2$, where $\phi(n)$ is the Euler ϕ-function counting integers a between 1 and $n - 1$ *inclusive* with $(a, n) = 1$. Then the problem reduces to showing that $\phi(n) > 2$ for $n \geq 7$, and this can be done fairly easily by looking at the well-known formula for $\phi(n)$: If $n = p_1^{e_1} \ldots p_r^{e_r}$ where the p_i are distinct primes and $e_i \geq 1$, then

$$\phi(n) = \prod_{i=1}^{r} p_i^{e_i - 1}(p_i - 1). \qquad ■$$

Solution 3: We will prove the stronger statement that any $n \geq 7$ can be expressed as $p + m$ where p and m are relatively prime integers greater than 1, *and p is prime*. If p is any prime less than $n - 1$, then $p > 1$, $n - p > 1$ and $p + (n - p) = n$. Moreover p and $n - p$ will be relatively prime, unless p divides $n - p$, or equivalently p divides n. Therefore it suffices to show that there is at least one prime less than $n - 1$ that does not divide n.

Suppose not. Then $2 \cdot 3 \cdot 5 \cdots \ell$ divides n, where ℓ is the largest prime less than $n-1$. In particular, $2 \cdot 3 \cdot 5 \cdots \ell \leq n$. Consider the odd integer $k := (3 \cdot 5 \cdots \ell) - 2$. Since $n \geq 7$, $k \geq 3 \cdot 5 - 2 > 1$, and hence k has an odd prime factor q. Then $q \leq k \leq n/2 - 2 < n-1$, so q appears in the product $3 \cdot 5 \cdots \ell = k+2$. Now q divides $k+2$ as well as k, so q divides 2, contradicting the fact that q is an odd prime.

Hence at least one of 2, 3, 5, ..., ℓ does not divide n, and that prime may be used as p. ■

Remark: Many students wrote an incorrect solution based on proving that there is a prime $p < n$ that does not divide n. This was proved by an argument similar to the one in the correct solution above: if there is no such prime, then $2 \cdot 3 \cdot 5 \cdots j$ divides n, where j is the largest prime less than n. Let q be a prime factor of $k := (2 \cdot 3 \cdot 5 \cdots j) - 1$. Then $q < n$, so q appears in the product $2 \cdot 3 \cdot 5 \cdots j$. Thus q divides $k+1$ as well as k, so q divides 1, a contradiction.

This argument does indeed prove that there is a prime $p < n$ not dividing n, but unfortunately it leaves open the possibility $p = n-1$. If $p = n-1$, then $n - p = 1$, so $p + (n-p)$ is not a partition of n of the desired form.

Solution 4: Another correct solution uses "Bertrand's postulate" that if $m > 3$, there exists a prime p with $m < p < 2m-2$. (This was first proved by Chebychev: see p. 373 of Hardy and Wright, *An Introduction to the Theory of Numbers*, 5th edition, Clarendon Press, 1979.) Applying this with $m = \lceil n/2 \rceil$, the smallest integer greater than or equal to $n/2$, produces a prime p with $n/2 < p < n-1$. Such p cannot divide n, since $p < n < 2p$. Hence as in the previous solution, the partition $n = p + (n-p)$ works. ■

Remark: It would *not* be enough to use the slightly weaker statement (often also called Bertrand's postulate) that for $m > 1$ there exists a prime p with $m < p < 2m$. If n is even, the choice $m = n/2$ does not rule out $p = n-1$. Choosing $m = n/2 - 1$ fixes this problem, but introduces the new problem that p might be $n/2$, which divides n.

2 We shall present two solutions. The second solution generalizes the problem.

Solution 1: Let k be the circumscribed circle of $\triangle ABC$. Let O be the center of k, and let O_1 be the center of circle k_1 (see Fig. 1.)

We claim that k_1 passes through O. Indeed, consider the dilation ρ with center A and coefficient $1/2$: it sends any point X in the plane the midpoint of segment AX. Since D and F are respectively the midpoints of AB and AC, then ρ sends B to D and C to F. Further, ρ does not move point A. In summary, ρ sends three points on circle k (A, B, C) to three points on circle k_1 $(A,\ D,\ F)$. But circles go to circles under dilations. Thus, ρ must have sent the circle k to the circle k_1. In particular, if A' is the diametrically opposite point to A on k, ρ must have sent A' to some point on k_1. On the other hand, since O is the center of k and hence the midpoint of AA', we see that ρ sends A' to O. Thus, point O lies on the circle k_1.

Using similar reasoning, we conclude that O also lies on circles k_2 and k_3. Therefore, the three circles k_1, k_2 and k_3 do indeed intersect in a common point, namely, the circumcenter O of $\triangle ABC$. ■

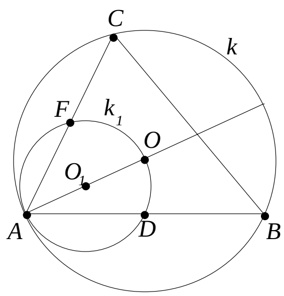


fig. 1

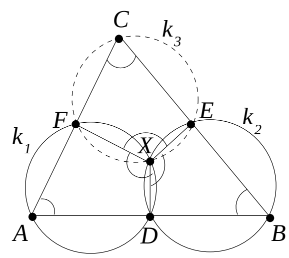


fig. 2

Solution 2: We claim that a more general statement is also true: The points D, E, F need not be the midpoints of the sides of $\triangle ABC$, but can be any points lying on the respective sides of the triangle. Then again circles k_1, k_2 and k_3 will intersect in a common point.

PROOF: Let k_1 and k_2 intersect in point X, other than D (see Fig. 2.) In k_1, $\angle DAF$ and $\angle FXD$ subtend opposite (complementary) arcs on k_1, and hence add up to 180°:

$$\angle DAF + \angle FXD = 180^\circ.$$

Similarly in k_2:

$$\angle DBE + \angle DXE = 180^\circ.$$

So, what is left for $\angle FXE$:

$$\angle FXE = 360^\circ - \angle FXD - \angle DXE = (180^\circ - \angle FXD) + (180^\circ - \angle DXE) = \angle FAD + \angle EBD.$$

The last sum is merely $180^\circ - \angle ACB$ (the angle sum in $\triangle ABC$ is 180°.) In summary, $\angle FXE = 180^\circ - \angle ACB$, or equivalently,

$$\angle FXE + \angle FCE = 180^\circ.$$

This is a necessary and sufficient condition for the points C, F, E and X to lie on the same circle. In other words, X also lies on k_3, and hence k_1, k_2 and k_3 indeed intersect in a common point, namely X.

The diligent reader will have noticed by now that the above solution implicitly assumes that the intersection point X of k_1 and k_2 is inside $\triangle FDE$. This does not need to be true, even in the special case when points D, E and F are midpoints of the sides of $\triangle ABC$. However, considering any other possible case leads to essentially the same solution.

For example, let point X be “below” side AB, i.e. C and X are on different sides of line AB, but X is still inside $\angle ACB$ (see Fig. 3.) Then in k_1 and k_2 the following angles subtend same arcs, and are therefore equal:

$$\angle FXD = \angle FAD, \quad \text{and} \quad \angle EXD = \angle EBD.$$

Hence $\angle FXE = \angle FAD + \angle EBD = 180^\circ - \angle ACB$. Again, $\angle FXE + \angle FCE = 180^\circ$, and hence points F, C, E and X lie on the same circle k_3.

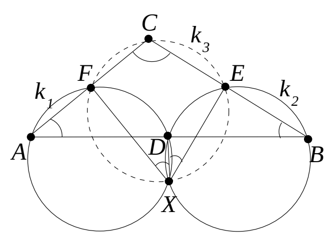


fig. 3

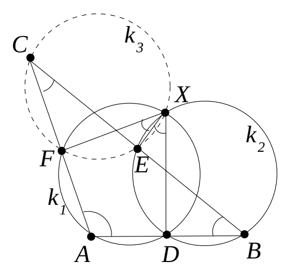


fig. 4

A similar situation occurs if X and A are on opposite sides of line BC, but X is still inside $\angle ABC$ (see Fig. 4.) We have from k_1 and k_2:

$$\angle FXD = 180^\circ - \angle FAD, \quad \text{and} \quad \angle EXD = \angle EBD,$$

$$\Rightarrow \ \angle FXE = \angle FXD - \angle EXD = 180^\circ - \angle FAD - \angle EBD = \angle ACE.$$

Thus, $\angle FXE = \angle FCE$, which implies that X lies also on circle k_3.

The reader may now check that in Figure 5 the shaded areas reflect positions of X for which the problem was already proved. It is easy to see that X cannot be inside $\triangle ADF$ or $\triangle DBE$

(afterall, these areas are contained entirely in k_1 or k_2!) It takes a simple argument to rule out the inside of $\triangle CFE$. (Show, for instance, that if X were there, then $\angle FXD + \angle EXD = 180^\circ + \angle C > 180^\circ$, which is ridiculous.) Finally, notice that k_1 cannot not pass through any of the regions α, β and γ, which eliminates them as possible locations for point X.

This exhausts all possibilities for the intersection point X of k_1 and k_2, and thus proves that X always lies on the third k_3 too. ■

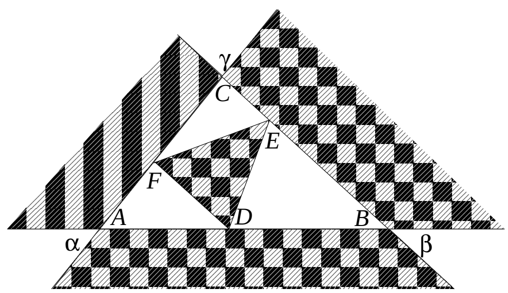


fig. 5

3 We shall present two solutions. The first solution uses "standard" inequality methods, while the second uses induction plus an "algorithmic" approach.

Solution 1: First we will prove a simple lemma involving only two variables: For all positive a, b,

$$(a^2+1)(b^2+1) \geq (ab+1)^2.$$

To see why this is true, multiply out, and after simplifying, we have

$$a^2+b^2 \geq 2ab.$$

This is equivalent to

$$a^2-2ab+b^2=(a-b)^2 \geq 0,$$

which of course is true (in fact, for any real numbers a and b).

Now we shall attack the problem. Multiplying both sides by $x_1x_2\cdots x_n$ produces the equivalent inequality

$$\left(x_1^2+1\right)\left(x_2^2+1\right)\cdots\left(x_n^2+1\right) \geq (x_1x_2+1)(x_2x_3+1)\cdots(x_nx_1+1).$$

Applying the lemma repeatedly yields

$$\begin{aligned}
\left(x_1^2+1\right)\left(x_2^2+1\right) &\geq (x_1x_2+1)^2\,, \\
\left(x_2^2+1\right)\left(x_3^2+1\right) &\geq (x_2x_3+1)^2\,, \\
&\vdots \\
\left(x_n^2+1\right)\left(x_1^2+1\right) &\geq (x_nx_1+1)^2\,.
\end{aligned}$$

Multiplying these yields the square of the desired inequality. ■

Solution 2 (sketch): We shall use induction. Even though the problem begins with $n = 2$, we can start by noting that for $n = 1$, the statement is merely the trivial

$$x_1 + 1/x_1 \geq x_1 + 1/x_1.$$

In general, suppose without loss of generality that x_1 is the largest among the given n numbers. The right-hand side products containing x_1 are: $(x_1 + 1/x_2)(x_n + 1/x_1)$. We claim that this product will not decrease if we swap the places of x_2 from the first multiple and x_1 from the second multiple, i.e.

$$(x_1 + 1/x_2)(x_n + 1/x_1) \leq (x_1 + 1/x_1)(x_n + 1/x_2).$$

This inequality is easy to prove: after a little algebra, it becomes

$$x_1x_2 + x_1x_n \leq x_1x_1 + x_2x_n,$$

which is equivalent to

$$(x_1 - x_n)(x_1 - x_2) \geq 0,$$

and this is true because x_1 was the largest number among the given n numbers.

Notice that after performing the "swap," we may cancel $(x_1 + 1/x_1)$ from both sides, and what we are left with is the same problem but for the $(n-1)$ numbers $x_2, x_3, \ldots x_n$. This completes the inductive step. ■

4 Let us define a k-configuration to be a finite set of points on the plane such that for each point P in the set, there are at least k points of the set 1 inch from P. Then the problem is asking us to show that there is a 2000-configuration with 3^{1000} points.

Notice that an equilateral triangle is a 2-configuration which has 3 points. Now, let A be a k-configuration with N points, and let T be an equilateral triangle with unit side length. We shall show that it is possible to "add" a A and T to create a $(k+2)$-configuration with $3N$ points:

Define the set S by

$$S = \{a + t | a \in A, t \in T\},$$

where we treat the points as vectors. In other words, S consists of the vector sums of every point in A with every point in T. Since A has N points, and T has 3 points, the set S will have $3N$ points as long as all of these sums are distinct. For the time being, let us assume that the sums are all distinct.

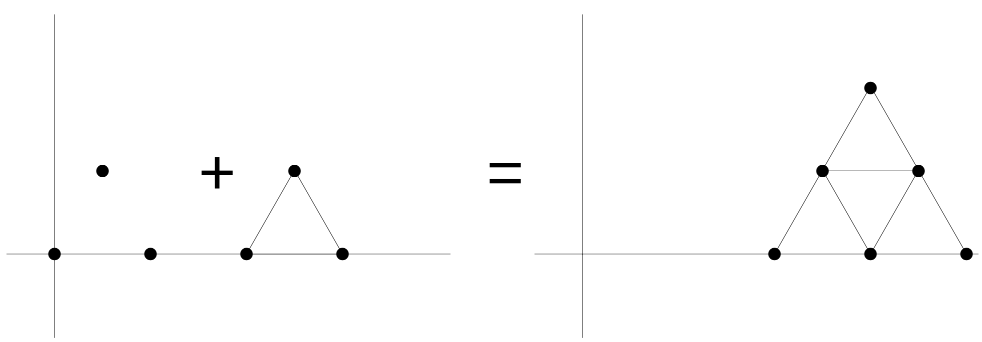
Now we will show that S is a $(k+2)$ configuration. Consider any point $a + t$ in S, where $a \in A$ and $t \in T$. Since A is a k-configuration, there are k points $a_1, \ldots, a_k \in A$ that are 1 inch away from a. Likewise, there are two points t_1, t_2 in T which are each 1 inch away from t. It is easy to check that the $k+2$ points

$$t + a_1, \ldots, t + a_k; a + t_1, a + t_2$$

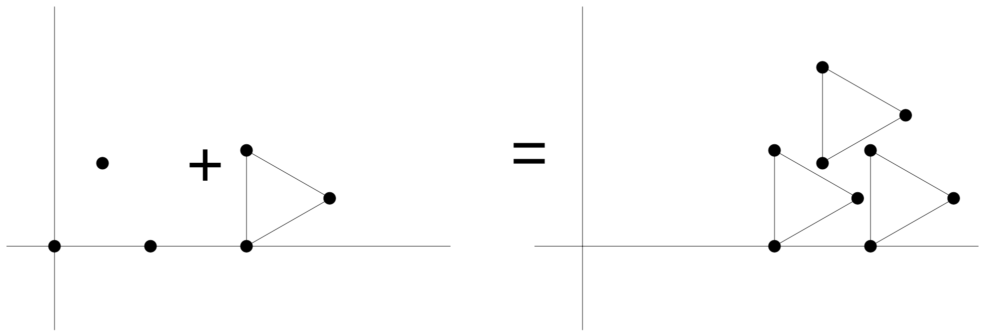
are each 1 inch away from $a + t$. Thus S is a $(k+2)$ configuration.

But how do we insure that that all $3N$ sums are distinct? The sums fail to be distinct only if there are pairs $a, a' \in A$ and $t, t' \in T$ with $a + t = a' + t'$ which in turn is true if and only if $a - a' = t' - t$. To ensure that this does not happen, it suffices to rotate one of the two sets (say, T) so that the slopes of all of the lines connecting all pairs of points in T do not equal any of the slopes in A (easy to do since there are finitely many points).

For example, in the following diagram, we attempt to "add" two equilateral triangles (the second triangle is outlined), but because of equal slopes, the sum contains only 6 points.



On the other hand, if we rotate the second triangle (in this case, by 30 degrees), the resulting sum contains 9 points (and you should check that this new set is indeed a 4-configuration).



Clearly, we can continue this summation process, adding additional copies of equilateral triangles (making sure to rotate so that no slopes are equal). For each triangle that we add, the new set will have three times as many points. Thus if we add 1000 triangles, we will get a set with 3^{1000} points which is a $2 + 2 + \cdots + 2 = 2000$-configuration.

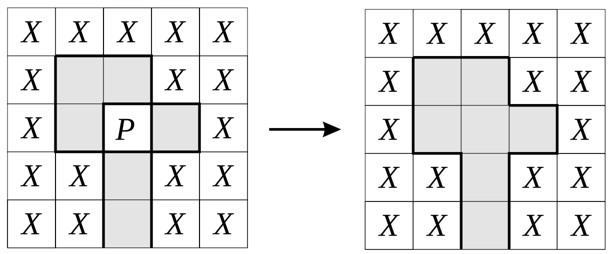
Remark: This construction can be generalized to show that if $s(k)$ denotes the *smallest* number of points possible for a k-configuration, then $s(k + m) \leq s(k)s(m)$. This may help the reader to investigate the deeper question: what is $s(k)$ for each k?

5 We shall present 3 solutions, all of which use the same basic idea of *monovariants*, quantities which vary monotonically (are either non-increasing or non-decreasing). The first solution develops the ideas in a leisurely fashion. The second solution is an ultra-compact version of the first, and the third solution is a terse argument which uses a different monovariant.

Solution 1: Assume that the squares have unit length. Consider, at any time during the game, the *perimeter* of the region of empty squares (this region may or may not be connected). For example, suppose that at some time 64 squares are empty. If the empty squares are packed together forming an 8×8 square, the perimeter will equal $4 \cdot 8 = 32$. On the other hand, if none of the 64 empty squares are adjacent to one another, the perimeter will equal $4 \cdot 64 = 256$ (each square has perimeter 4).

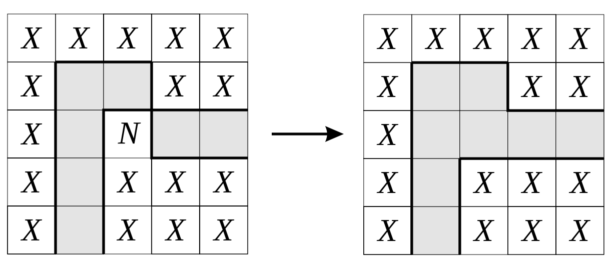
At the start of the game, there are $k = 59$ empty squares. At this point the the perimeter is at most $4k$. Now consider what happens as different coins are removed.

In order for a penny to be removed, it has to be surrounded on all 4 sides by empty squares. The following diagram illustrates the situation. The shaded squares are empty, the squares containing "X" are occupied by arbitrary coins, and "P" of course denotes a penny. The thick lines indicate the perimeter.



Before the removal, the perimeter included the boundary of the square containing the penny. After the removal, this boundary is gone. Thus, whenever a penny is removed, the perimeter decreases by 4.

If a nickel is removed, it is possible that it was originally surrounded on all four sides by empty squares as above, in which case the perimeter will decrease by 4. However, the nickel may have been adjacent to only three empty squares as seen below.



In this case, the perimeter decreases by 2. Thus, no matter what, after a nickel is removed, the perimeter decreases by at least 2.

By similar reasoning (draw diagrams!) we conclude that if a dime is removed, the perimeter does not increase (it also could decrease by 2 or 4), and when a quarter is removed, the perimeter may increase, but by at most 2 (it also could not change or decrease by 2 or 4).

Now we will show that it is impossible for the board to evolve so that no coins are left. Note that this would cause the perimeter to equal $4 \cdot 20 = 80$. Suppose that at the start, p pennies, n nickels, d dimes, and q quarters were removed (so $p + n + d + q = k$). Let $t = 10$. If the board evolved so that all the remaining coins were removed, then $t^2 - p$ pennies would be removed, decreasing the perimeter by exactly $4(t^2 - p)$. Likewise, $t^2 - n$ nickels would be removed, but this will decrease the perimeter by at least $2(t^2 - n)$. The $t^2 - d$ dimes that would be removed decrease the perimeter by at least zero. And finally, $t^2 - q$ quarters are removed and these could make the perimeter *increase* by *at most* $2(t^2 - q)$. Recall that the starting value for the perimeter is at most $4k$. Thus, if all coins were removed, the final value of the perimeter would be at most

$$4k - 4(t^2 - p) - 2(t^2 - n) + 2(t^2 - q) = 4k - 4t^2 + 4p + 2n - 2q.$$

But $4p + 2n - 2q$ is certainly less than or equal to $4(p + n + d + q) = 4k$, so the final value of the perimeter is at most

$$4k - 4t^2 + 4k = 8k - 4t^2 = 8 \cdot 59 - 4 \cdot 10^2 = 72,$$

Contradicting the fact that the perimeter must equal 80. ■

Solution 2: Define P to be perimeter of empty region (as defined in the previous solution) and let q, d, n be respectively the number of quarters, dimes, and nickels on the board at a given time. Then the quantity

$$P - 4q - 2d + 2n$$

is a monovariant: it is *non-increasing* (no matter how the coins are legally removed). The initial value of this quantity is at most 72, yet the value of the empty board is 80. Hence the board will never be empty. ■

Solution 3: For simplicity, give each coin the value $1, 2, 3, 4$ (pennies, nickels, dimes, quarters) according to the minimum number of adjacent squares that are required for their removal. If x is a configuration of coins let $f(x)$ denote the sum of the coins plus the number of pairs of adjacent empty squares on the board. If x' is a configuration obtained from x by (legally) removing some coin, then $f(x') \geq f(x)$ by the rules (e.g., removing a dime decreases the total coin value by 3 but increases the number of orthogonally adjacent pairs by at least 3). In other words, $f(x)$ is a monovariant. A simple count shows that $f(\text{empty}) = 760$ and

$$f(\text{initial}) = 764 = 100 \cdot 1 + 100 \cdot 2 + 100 \cdot 3 + 41 \cdot 4,$$

since the smallest initial $f()$ is obtained by removing 59 pennies (and not creating any adjacent squares). Since we can't move upwards from 764 and end up at 760 the desired sequences of moves doesn't exist. ■