4th Bay Area Mathematical Olympiad

February 26, 2002

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

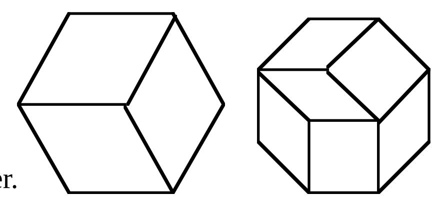
Please label all pages that you submit for grading with your identification number in the upper right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. In particular, problems 4 and 5 are quite difficult. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

1 Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

2 In the illustration, a regular hexagon and a regular octagon have been tiled with rhombuses. In each case, the sides of the rhombuses are the same length as the sides of the regular polygon.



(a) Tile a regular decagon (10-gon) into rhombuses in this manner.

(b) Tile a regular dodecagon (12-gon) into rhombuses in this manner.

(c) How many rhombuses are in a tiling by rhombuses of a 2002-gon? Justify your answer.

Make your drawings on the sheet of decagons and dodecagons provided. Clearly indicate which drawings you would like to have graded.

3 A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. The winner is the player who makes a move that causes all stacks to be of height 1 or 2. For which starting values of n does the player who goes first win, assuming best play by both players?

Please turn over for problems #4 and #5.

4 For $n \geq 1$, let a_n be the largest odd divisor of n, and let $b_n = a_1 + a_2 + \cdots + a_n$. Prove that $b_n \geq \dfrac{n^2+2}{3}$, and determine for which n equality holds. For example,

$$a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 1, a_5 = 5, a_6 = 3,$$

thus

$$b_6 = 1+1+3+1+5+3 = 14 \geq \frac{6^2+2}{3} = 12\frac{2}{3}.$$

5 Professor Moriarty has designed a "prime-testing trail." The trail has 2002 stations, labeled 1, ..., 2002. Each station is colored either red or green, and contains a table which indicates, for each of the digits 0, ..., 9, another station number. A student is given a positive integer n, and then walks along the trail, starting at station 1. The student reads the first (leftmost) digit of n, and looks this digit up in station 1's table to get a new station location. The student then walks to this new station, reads the second digit of n and looks it up in this station's table to get yet another station location, and so on, until the last (rightmost) digit of n has been read and looked up, sending the student to his or her final station. Here is an example that shows possible values for some of the tables. Suppose that $n = 19$:

Station	0	1	2	3	4	5	6	7	8	9
1 (red)	15	29	314	16	2002	97	428	1613	91	24
⋮										
29 (red)	98	331	1918	173	15	41	17	631	1211	1429
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1429 (green)	7	18	31	43	216	1709	421	53	224	1100

Using these tables, station 1, digit 1 leads to station 29; station 29, digit 9 leads to station 1429; and station 1429 is green.

Professor Moriarty claims that for any positive integer n, the final station (in the example, 1429) will be green if and only if n is prime. Is this possible?

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2002 Awards Ceremony**, which will be held at San Jose State University from 11–2 on Sunday, March 10. This event will include lunch (free of charge), a mathematical talk by Professor Joseph Gallian of the University of Minnesota, Duluth, and the awarding of dozens of prizes, worth thousands of dollars. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.

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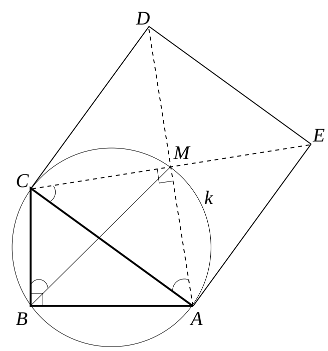
Problems and Solutions

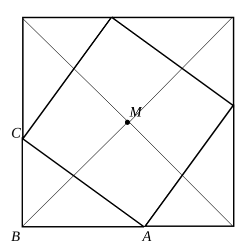
1 Let ABC be a right triangle with right angle at B. Let $ACDE$ be a square drawn exterior to triangle ABC. If M is the center of this square, find the measure of $\angle MBC$.

Solution 1:



Note that triangle MCA is a right isosceles triangle with $\angle ACM = 90^\circ$ and $\angle MAC = 45^\circ$. Since $\angle ABC = 90^\circ$, there is a circle k with diameter AC which also passes through points B and C. As inscribed angles, $\angle MAC = \angle ABC$, thus the measure of $\angle MBC = 45^\circ$. ■

Solution 2: Place 3 copies of triangle ABC on the square as shown below.

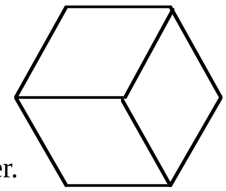


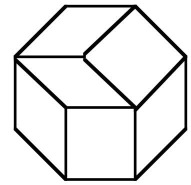
Clearly the new diagram is a large square (which can be proven easily by looking at the angles of the copies of the triangle). The diagonals of this large square meet at M. By symmetry, $\angle MBC = 45^\circ$. ■

Solution 3: Place triangle ABC on coordinate axes so that $A = (0, 2c)$, $B = (0, 0)$, $C = (2a, 0)$. Let N be the midpoint of hypotenuse AC and draw line ℓ through N parallel to BC which meets AB at F. Drop a perpendicular from M to BC which meets ℓ at G. It is evident that triangles AFN and NGM are congruent.

Thus $NG = AF = c$ and $MG = FN = a$. Consequently, the coordinates of M are $(a, c) + (c, a) = (a + c, a + c)$, so the line from M to the origin (at B) has a slope of 1. ■

2 In the illustration, a regular hexagon and a regular octagon have been tiled with rhombuses. In each case, the sides of the rhombuses are the same length as the sides of the regular polygon.



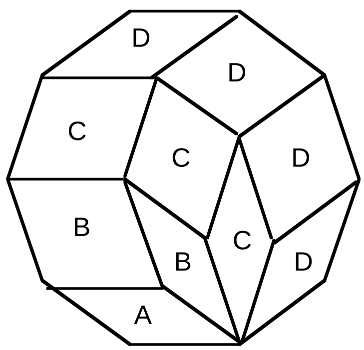


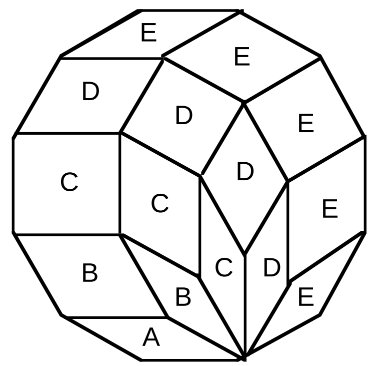
(a) Tile a regular decagon (10-gon) into rhombuses in this manner.

(b) Tile a regular dodecagon (12-gon) into rhombuses in this manner.

(c) How many rhombuses are in a tiling by rhombuses of a 2002-gon? Justify your answer.

Solution: For parts (a) and (b), see the tilings below of a decagon and a dodecagon with rhombuses.





(c) **Solution 1**: We shall describe an algorithm for tiling any regular $2n$-gon. First construct a rhombus A using two sides of a regular $2n$-gon. On the two interior sides construct two rhombuses B using a side adjacent and clockwise to the exterior sides of A. On the three interior sides of the B rhombuses construct three rhombuses C using the next side of the $2n$-gon going clockwise. After the $(n-2)^{nd}$ step an outer layer of $n-2$ rhombuses is in place. To see that the $(n-1)^{st}$ step completes the $2n$-gon note that the $(n+1)^{st}$ side after the first side of A is parallel and equal in length to the first side of A and has the same length as the nth side of the $2n$-gon. Therefore it is a rhombus. In the same way, the remaining $n-1$ sides of the $2n$-gon form rhombuses with segments parallel to the n^{th} side. The total number of rhombuses is the sum of the first $n-1$ integers. Then it is the case that a regular 2002-gon can be tiled with $1+2+3+\cdots+999+1000 = 1000(1001)/2 = 500(1001) = 500{,}500$ rhombuses. ■

Solution 2: We will prove the slightly more general fact that any equilateral $2n$-gon with opposite sides parallel (we will call such polygons “EP-gons”) can be tiled with $1+2+\cdots+n-1$ rhombuses. We proceed by induction on n. Certainly the result is true for $n=2$, since a 4-sided EP-gon is simply a single rhombus.

For the inductive step, suppose we have $(2n+2)$-sided EP-gon. We will use the leftmost figure above for reference; thus in this case $n=4$. Select one side of this polygon (we will use the “bottom” side in the picture, one of the sides of rhombus A). Then draw $n-1$ lines parallel to this side with lengths equal to this side, each line starting at a vertex of the polygon. Then draw n lines parallel to the sides of the polygon; when we are done we will have drawn a “shell” of n rhombuses on the left of the polygon (rhombus A and the leftmost rhombuses B, C, and D). We claim that the remainder of the $(2n+2)$-gon (the non-tiled region) is a $2n$-sided EP-gon. To see this, note that the untiled region consists of n sides of the right half of the original $(2n+2)$-gon, plus n new sides, each drawn parallel (and equal in length) to a side of the left half of the original $(2n+2)$-gon. It is as if we rigidly slided the left half of the original $(2n+2)$-gon to the right, so that the top and bottom sides of the original polygon disappeared. Thus in this new region, opposite sides are parallel. By the inductive hypothesis, this region can be tiled with $1+2+\cdots+(n-1)$ rhombuses. Thus the original $(2n+2)$-gon can be tiled by $1+2+3+\cdots+(n-1)+n$ rhombuses. ■

3 A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. The winner is the player who makes a move that causes all stacks to be of height 1 or 2. For which starting values of n does the player who goes first win, assuming best play by both players?

Solution 1: Player 1 wins if and only if $n = 3$ or n is even; player 2 wins for all odd $n > 3$. We can easily check this for the first few cases, say up to $n = 6$, and then we can proceed by induction.

- If $n > 6$ is even, player 1 creates a pile of size 1 and a pile of size $n-1$. Since $n-1$ is odd, player 1 will win by the inductive hypothesis (since player 1 is now going second and the "starting position" has an odd number of pennies).
- If $n \geq 7$ is odd, player 1 will create one odd and one even pile. Player 2 can then divide the even pile into two odd piles. Continuing in this way, player 2 can always answer player 1's move and present player 1 with only odd piles. When the piles reach size 1, they are irrelevant. The critical value is size 3: the only way that player 2 can lose is if player 2 presents player 1 with a single 3-pile (and many 1-piles). But for this to happen, player 1 would have to have produced either a single 2-pile and one 3-pile, or a single 4-pile. In either case, player 2 wins on the next move by reducing, in the first case, the 3-pile to a 1- and 2-pile, and in the second case, breaking the 4-pile into two 2-piles. In sum, player 2's winning strategy is to always produce only odd piles unless this will produce all 1's and a single 3; in which case the "terminal" strategy above is employed.

■

Solution 2 (Sketch): At each stage of the game, let S be the sum of one less than each pile height. For example, if the piles are 8, 6, 2, 2, 1, then $S = 7+5+1+1+0$. Observe that S is also equal to the total number of pennies minus the number of piles, and thus, S always decreases by 1 each turn. We shall analyze the parity of S.

The only way a player will win is if on their turn, one pile has 3 or 4 pennies, with the other piles (if there are any) of height 2 or 1. We call such a position "penultimate." For example, 4, 2, 2, 2, 1 is penultimate (and the winning move will split the 4 into two piles).

If a position is not penultimate, but can be turned into a penultimate position in one move, we call it "antepenultimate." For example, 4, 3, 2, 2, 2, 1 is antepenultimate (if the 4 is split). Notice that an antepenultimate position doesn't always have to become a penultimate position in one move. For example, with the position above, instead of splitting the 4, we could split one of the 2-piles.

The antepenultimate positions fall into 2 cases:

- A 5 or 6, and the rest (if any) are 2s and 1s.
- Two piles that are 3 or 4, and the rest (if any) are 2s and 1s.

We make two observations:

- The number of 1's are irrelevant (since they cannot be changed, and also they do not alter the value of S).
- Without loss of generality, the number of 2s in a pile (if there are any) is either 1 or 2. If a position had more than two 2's, they can be removed and it would not change the parity of S, nor would it affect who wins the game (since the only thing that can be done with a 2-pile is either to leave it alone or split it into two 1-piles)

Thus there are only a few cases to check, and in all of them, we discover that

If a position is antepenultimate and S is odd, then the player who has this position will win. If a position is antepenultimate and S is even, then the player who has this position will lose.

For example, 5, 2, 1 is antepenultimate and $S = 5$. The player splits the 2, and then her opponent has no choice but to split the 5, handing the penultimate position to the first player, who wins.

Now we can use this analysis to look at what happens from the very start of the game.

- If $n = 3$ or $n = 4$, the first player already has a penultimate position and wins.
- If $n = 5$, then the starting position is antepenultimate and S is even. Clearly the second player wins.
- If $n = 6$, then the starting position is antepenultimate and S is odd. Clearly the first player wins.
- If $n > 7$, then the starting position is not antepenultimate. If S is odd to start, it will always be odd when it is player 1's turn (since S drops by 1 each time). So at some point, player 1 will either get the penultimate position and will win, or will get the antepenultimate position with odd S, and will win. By similar reasoning, if n is odd, S starts out even, and player 1 will lose.

■

4 For $n \geq 1$, let a_n be the largest odd divisor of n, and let $b_n = a_1 + a_2 + \cdots + a_n$. Prove that $b_n \geq \dfrac{n^2+2}{3}$, and determine for which n equality holds. For example,

$$a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 1, a_5 = 5, a_6 = 3,$$

thus

$$b_6 = 1 + 1 + 3 + 1 + 5 + 3 = 14 \geq \frac{6^2+2}{3} = 12\frac{2}{3}.$$

Solution: Inspecting a few values leads one to the correct guess that equality holds if and only if n is a power of 2. The idea is to use induction by "doubling" to prove the result. When $n = 1$, we have $1 = 1$ as desired. Now suppose $n \geq 2$. If $n = 2k$ with $k \geq 1$, then

$$\begin{aligned} b_n &= (a_1 + a_3 + \cdots + a_{2k-1}) + (a_2 + a_4 + \cdots + a_{2k}) \\ &= 1 + 3 + \cdots + (2k-1) + (a_1 + a_2 + \cdots + a_k) \\ &= k^2 + b_k \\ &\geq k^2 + (k^2+2)/3 \quad \text{(by the inductive hypothesis)} \\ &= (n^2+2)/3, \end{aligned}$$

with equality if and only if k is a power of 2, i.e., if and only if n is a power of 2. If instead $n = 2k+1$ with $k \geq 1$, then

$$\begin{aligned} b_n &= (a_1 + a_3 + \cdots + a_{2k+1}) + (a_2 + a_4 + \cdots + a_{2k}) \\ &= 1 + 3 + \cdots + (2k+1) + (a_1 + a_2 + \cdots + a_k) \\ &= (k+1)^2 + b_k \\ &\geq (k+1)^2 + (k^2+2)/3 \quad \text{(by the inductive hypothesis)} \\ &= \frac{(2k+1)^2+2}{3} + \frac{2k+2}{3} \\ &> (n^2+2)/3 \end{aligned}$$

so strict inequality holds, as desired. ∎

5 Professor Moriarty has designed a "prime-testing trail." The trail has 2002 stations, labeled $1, \ldots, 2002$. Each station is colored either red or green, and contains a table which indicates, for each of the digits $0, \ldots, 9$, another station number. A student is given a positive integer n, and then walks along the trail, starting at station 1. The student reads the first (leftmost) digit of n, and looks this digit up in station 1's table to get a new station location. The student then walks to this new station, reads the second digit of n and looks it up in this station's table to get yet another station location, and so on, until the last (rightmost) digit of n has been read and looked up, sending the student to his or her final station. Here is an example that shows possible values for some of the tables. Suppose that $n = 19$:

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Using these tables, station 1, digit 1 leads to station 29; station 29, digit 9 leads to station 1429; and station 1429 is green.

Professor Moriarty claims that for any positive integer n, the final station (in the example, 1429) will be green if and only if n is prime. Is this possible?

Solution: No, this is impossible. Suppose on the contrary that such a trail has been designed. Since there are infinitely many primes, we can choose a prime p with more than 2002 decimal digits . When we test p on the trail, we will visit some station more than once. Write p as $10^N A + 10^m B + C$ so that the station after the digits of A is the same as after the digits of B. Then we may reach the same station by running through the digits of A, then running through the digits of B any number of times, then running through the digits of C.

That is, for all $k \geq 0$, the numbers

$$C + 10^m B\left(1 + 10^{N-m} + 10^{2(N-m)} + \cdots + 10^{k(N-m)}\right) + 10^{(k+1)(N-m)+m} A$$

$$= C - \frac{10^m B}{10^{N-m}-1} + 10^{(k+1)(N-m)}\left(10^m A + \frac{10^m B}{10^{N-m}-1}\right)$$

all lead to the same terminal station, and so must all be prime. For simplicity, write this expression as $p_k = r + t^k s$, and choose k large enough so that p_k does not divide the denominators of r, s, t. Then $t^k \equiv t^{k+p_k-1} \pmod{p_k}$ by Fermat's little theorem, so p_{k+p_k-1} is divisible by p_k and hence is not prime, contradiction. ∎

NOTE. A device conforming to the description in the problem is known in computer science as a *finite-state automaton*.

No one got full credit for this problem. The best partial solution was due to Phillip Sung, of Saratoga HS, who argued that if there existed a "repunit" prime with more than 2002 digits (a repunit prime is prime consisting of repeated 1s; for example, 11), then we would have a contradiction. Unfortunately, the longest repunit prime known today has 1051 digits. It has been conjectured that there are repunit primes with over 30,000 digits, but this has not been proven. We leave as exercises for the reader to prove that all repunit primes have a prime number of digits, and that if there was a repunit prime with more than 2002 digits, that Phillip would have had a complete solution!