12th Bay Area Mathematical Olympiad

February 23, 2010

Problems (with Solutions)

1 We write $\{a,b,c\}$ for the set of three different positive integers a, b, and c. By choosing some or all of the numbers a, b and c, we can form seven nonempty subsets of $\{a,b,c\}$. We can then calculate the sum of the elements of each subset. For example, for the set $\{4,7,42\}$ we will find sums of 4, 7, 42, 11, 46, 49, and 53 for its seven subsets. Since 7, 11, and 53 are prime, the set $\{4,7,42\}$ has exactly three subsets whose sums are prime. (Recall that prime numbers are numbers with exactly two different factors, 1 and themselves. In particular, the number 1 is *not* prime.)

What is the largest possible number of subsets with prime sums that a set of three different positive integers can have? Give an example of a set $\{a,b,c\}$ that has that number of subsets with prime sums, and explain why no other three-element set could have more.

Solution:

The answer is five.

For example, the set $\{2,3,5\}$ has 2, 3, 5, 5, 7, 8, and 10 as its sums, and the first five of those are prime. If you're worried about 5 appearing twice in that list, then try $\{2,3,11\}$ which has 2, 3, 11, 5, 13, 14, and 16 as its subsets' sums, so now we see five different primes.

No set can have more than five prime subset sums because for any set $\{a,b,c\}$:

- If the set contains three even numbers, then clearly it can have only one prime subset sum, namely 2 if it is in the set.
- If the set contains two even numbers a and b then a, b, and $a+b$ are all even. Since they are distinct positive integers, only one of a and b can be equal to 2, and $a+b>2$, so we have at least two non-prime sums and thus at most five prime subset sums.
- If the set contains one even number a then a, $b+c$, and $a+b+c$ are all even. Again, only a can be equal to 2 and thus prime, so we have at least two non-prime sums as in the previous case.
- If the set contains zero even numbers (and thus three odd numbers) then $a+b$, $a+c$, and $b+c$ are all even, and since the numbers are distinct positive integers then none of these three sums can equal 2, so none of those are prime. Thus the set has at most four prime subset sums.

In any case, the maximum number of prime subset sums is five. ∎

There are various ways to shorten the above argument. For example, once you have found a set with five prime subset sums, you can check whether 6 is possible by looking at two cases: either two or three of the elements of the set have to be prime.

2 A *clue* "k digits, sum is n" gives a number k and the sum of k distinct, nonzero digits. An *answer* for that clue consists of k digits with sum n. For example, the clue "Three digits, sum is 23" has only one answer: 6,8,9. The clue "Three digits, sum is 8" has two answers: 1,3,4 and 1,2,5.

If the clue "Four digits, sum is n" has the largest number of answers for any four-digit clue, then what is the value of n? How many answers does this clue have? Explain why no other four-digit clue can have more answers.

Solution: The sum of 20 has 12 answers, and this is the largest number of answers for any four-digit clue.

We could simply list all the possible sets of four digits and then count. There are 126 such sets.

Alternatively, define $A(s,n,k)$ to be the number of options with sum s using exactly n digits whose largest digit is less than or equal to k. Then the question is to find the maximum of $A(s,4,9)$ for all values of s.

By symmetry, since we can replace each digit d with $10-d$, we know that $A(s,4,9) = A(40-s,4,9)$, so we only need to investigate values of s from the minimum, 1+2+3+4 = 10, through 20. (This also implies we only need to list the 69 sets of four digits whose sum is less than or equal to 20 in order to prove that 20 has the most answers. In fact, we can use even fewer than that, since by adding 1 to the largest digit we can see that $A(s,n,9) \le A(s+1,n,9)$ as long as there are no ways of writing s using the digit 9; for sums of four digits this shows we only need to investigate sums of 15 through 20.)

To compute $A(s,n,k)$ in general, we note that any sum must either use a digit equal to k or not. If there is a digit equal to k, then there are $A(s-k,n-1,k-1)$ ways to finish the sum. If there is no digit k, then there are $A(s,n,k-1)$ ways to finish the sum. Thus, $A(s,n,k) = A(s-k,n-1,k-1) + A(s,n,k-1)$.

We also know that $A(s,n,k)$ is 0 in a lot of cases, including any where $k < n$, and $A(s,1,k)$ is equal to 1 when $0 < s < k$ and 0 otherwise, because we must have one digit that equals s.

Thus, we can fill in the following tables, beginning with $n = 2$, and then $n = 3$, and then finally $n = 4$.

$n=2$	$s=3$	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=2$	1														
$k=3$	1	1	1												
$k=4$	1	1	2	1	1										
$k=5$	1	1	2	2	2	1	1								
$k=6$	1	1	2	2	3	2	2	1	1						
$k=7$	1	1	2	2	3	3	3	2	2	1	1				
$k=8$	1	1	2	2	3	3	4	3	3	2	2	1	1		
$k=9$	1	1	2	2	3	3	4	4	4	3	3	2	2	1	1

$n=3$	$s=6$	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$k=3$	1																		
$k=4$	1	1	1	1															
$k=5$	1	1	2	2	2	1	1												
$k=6$	1	1	2	3	3	3	3	2	1	1									
$k=7$	1	1	2	3	4	4	5	4	4	3	2	1	1						
$k=8$	1	1	2	3	4	5	6	6	6	6	5	4	3	2	1	1			
$k=9$	1	1	2	3	4	5	7	7	8	8	8	7	7	5	4	3	2	1	1

$n=4$	$s=10$	11	12	13	14	15	16	17	18	19	20
$k=4$	1										
$k=5$	1	1	1	1							
$k=6$	1	1	2	2	3	2	2	1	1		
$k=7$	1	1	2	3	4	4	5	4	4	3	2
$k=8$	1	1	2	3	5	5	7	7	8	7	7
$k=9$	1	1	2	3	5	6	8	9	11	11	12

We see that 20 has 12 answers, while 19 and 18 have only 11 answers (and similarly 21 and 22 also have 11 answers), and the remaining numbers have even fewer answers.

■

3 Suppose a,b,c are real numbers such that $a+b \geq 0$, $b+c \geq 0$, and $c+a \geq 0$. Prove that

$$a+b+c \geq \frac{|a|+|b|+|c|}{3}.$$

(Note: $|x|$ is called the *absolute value* of x and is defined as follows. If $x \geq 0$ then $|x| = x$; and if $x < 0$ then $|x| = -x$. For example, $|6| = 6$, $|0| = 0$ and $|-6| = 6$.)

Solution: The inequality $b+c \geq 0$ gives $a+b+c \geq a$. On the other hand, adding up the other two given inequalities yields $(a+b)+(c+a) \geq 0$, resulting in $a+b+c \geq -a$. Since $|a| = a$ or $-a$, we have in any case that

$$a+b+c \geq |a|.$$

Similarly

$$a+b+c \geq |b|$$

$$a+b+c \geq |c|$$

Now adding these three inequalities and dividing by 3 yields the desired inequality. ■

Alt Solution 1: The previous solution used the symmetry of a, b, and c. We can also use that symmetry to assume without loss of generality that $a \geq b \geq c$.

If b and c are both negative, then so is $b+c$, which contradicts the given information. So there can be at most one negative value among the three, which with our ordering must be c.

In the case where a, b, and c are all positive or 0, then the positive (or zero) number $x = a+b+c$ is greater than or equal to $x/3$.

Otherwise, since we have assumed that c is the least of the three, c is negative while a and b are not. Then $a+b+c = a+b-|c| \geq a$ since $b+c \geq 0$ tells us that $b \geq -c = |c|$. On the other hand $\frac{|a|+|b|+|c|}{3} = \frac{a+b-c}{3} \leq a$ since the average of three numbers is less than or equal to the greatest of the numbers. By transitivity we have $a+b+c \geq \frac{|a|+|b|+|c|}{3}$. ■

Alt Solution 2: The case where a, b, and c are all positive or zero can be handled as before. For the case where $a,b \geq 0$ and $c < 0$, $a+b+c-\frac{|a|+|b|+|c|}{3} = a+b+c-\frac{a+b-c}{3} = \frac{2a+2b+4c}{3} = \frac{2(a+b)+2(b+c)}{3} \geq 0$. ■

4 Place eight rooks on a standard 8×8 chessboard so that no two are in the same row or column. With the standard rules of chess, this means that no two rooks are attacking each other. Now paint 27 of the remaining squares (not currently occupied by rooks) red.

Prove that no matter how the rooks are arranged and which set of 27 squares are painted, it is always possible to move some or all of the rooks so that:

- All the rooks are still on unpainted squares.
- The rooks are still not attacking each other (no two are in the same row or same column).
- At least one formerly empty square now has a rook on it; that is, the rooks are not on the same 8 squares as before.

Solution: Look at the $\binom{8}{2} = 28$ pairs of rooks. (Ignore the coloring for now.) Each pair of rooks determines a pair of empty squares in the usual way: take the other two vertices of the rectangle (with sides parallel to the edge of the chessboard, of course) having our given pair of rooks as two vertices. (The opposite vertices will be empty since the rooks are non-attacking.) Furthermore, a given empty square is determined in this way by exactly one pair of rooks—the unique rooks in the same row and column as the given empty square.

Now by the Pigeonhole Principle, since there are 28 pairs of rooks and only 27 painted squares, one of the pairs of rooks determines a pair of empty squares which are both uncolored. Move these two rooks onto the empty squares instead, and you're done. ■

Many people wanted to put all the rooks on the diagonal, "without loss of generality". It needs to be shown that no generality is lost in doing this. One clever approach is to number the rooks 1 through 8, and then number the rows and columns according to which rook is in them. That way every square has coordinates (x, y), which is to say the square is in the column with rook x and the row with rook y. Thus all the rooks have coordinates (x, x) so they can be thought of as being on the diagonal without even having to move them!

5 All vertices of a polygon P lie at points with integer coordinates in the plane, and all sides of P have integer lengths. Prove that the perimeter of P must be an even number.

Solution: Travel around the polygon in one orientation (say, counterclockwise), and let the vertices so visited be $x_1, x_2, \ldots, x_n$. Define $\Delta x_i = x_{i+1} - x_i$, for $i = 1, 2, \ldots, n-1$ and $\Delta x_n = x_1 - x_n$. Define Δy_i in a similar way. Then the perimeter is equal to

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

Since each length $\sqrt{\Delta x_i^2 + \Delta y_i^2}$ is an integer, then for each i, either both Δx_i and Δy_i are even or exactly one is odd (they cannot both be odd using a mod-4 analysis). In the first case, we get an even length, and in the second case, we get an odd length.

So we need to show that the second case occurs an even number of times. This follows from the fact that

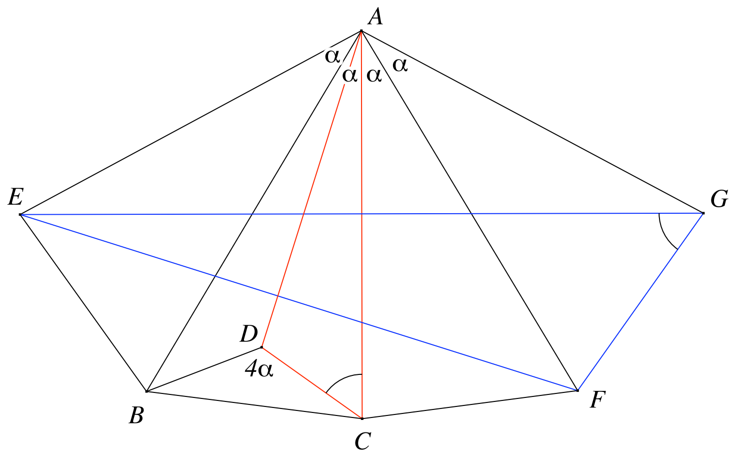
$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} \Delta y_i = 0.$$

Since 0 is even, there are an even number of odd Δx_is and there are an even number of odd Δy_is. ■

6 Acute triangle ABC has $\angle BAC < 45^\circ$. Point D lies in the interior of triangle ABC so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that lines AD and EF are perpendicular.

Solution: Based on the brilliancy award-winning solution by Evan O'Dorney.

Begin by reflecting C over AF to point G, as shown in the diagram.



As usual we denote $\alpha = \angle BAC$, $\beta = \angle ABC$, and $\gamma = \angle ACB$.

In order to prove that $AD \perp EF$, we will show that $\triangle ADC \sim \triangle EFG$ and that $AC \perp EG$.

To begin with the easier part, by reflection the four angles marked α are congruent, and $AE = AC = AG$. Thus AC is the angle bisector of isosceles triangle AEG and therefore $AC \perp EG$.

Next, to take a first step toward showing that $\triangle ADC \sim \triangle EFG$, we see that $BD = CD$ and $\angle BDC = 4\alpha$, so triangle AEG and BDC are similar isosceles triangles with base angle $90^\circ - 2\alpha$. (This is why the reflection of C over AF was a brilliant idea!) Furthermore, $\angle AGF = \angle ACF = \angle ACB = \gamma$ by reflection. Combining these facts, we have $\angle DCB = 90^\circ - 2\alpha$, and thus $\angle ACD = \gamma - (90^\circ - 2\alpha) = \angle EGF$.

Again using the similar triangles EAG and BDC,

$$\frac{AG}{EG} = \frac{DC}{BC},$$

and because of the reflection $\triangle ABC \cong \triangle AFG$,

$$\frac{FG}{AG} = \frac{BC}{AC}.$$

Multiplying these two equations gives $\dfrac{FG}{EG} = \dfrac{DC}{AC}$.

Consequently $\triangle ADC \sim \triangle EFG$ by side-angle-side similarity.

Since $AC \perp EG$, and both $\angle DAC$ and $\angle FEG$ are oriented in the same direction, the transformation that takes $\triangle ADC$ to $\triangle EFG$ is a 90° rotation, combined with some dilations and/or translations. This transformation also takes AD to EF, which implies that these lines are perpendicular.

■

Additional notes: It is interesting that this solution did not make use of the fact that, with O the circumcenter of $\triangle ABC$, we have $\angle BOC = 2\alpha$, and thus D is the circumcenter of $\triangle BOC$.

There are many other sets of similar triangles that could be used for an argument like this, based on side-angle-side similarity. Most if not all of them require adding more points to the diagram than just point G, which is yet more evidence of the brilliance of this solution.

7 Let a, b, c, and d be positive real numbers satisfying $abcd = 1$. Prove that

$$\frac{1}{\sqrt{\frac{1}{2}+a+ab+abc}}+\frac{1}{\sqrt{\frac{1}{2}+b+bc+bcd}}+\frac{1}{\sqrt{\frac{1}{2}+c+cd+cda}}+\frac{1}{\sqrt{\frac{1}{2}+d+da+dab}} \geq \sqrt{2}.$$

Solution: Let

$$\begin{aligned} S_a &= a+ab+abc \\ S_b &= b+bc+bcd \\ S_c &= c+cd+cda \\ S_d &= d+da+dab. \end{aligned}$$

Notice that $1+S_a = \frac{1}{2}+\left(\frac{1}{2}+S_a\right) \geq 2\sqrt{\frac{1}{2}\cdot\left(\frac{1}{2}+S_a\right)} = \sqrt{2}\cdot\sqrt{\frac{1}{2}+S_a}$. Using similar relations for S_b, S_c, and S_d we see that the left-hand side of the required inequality is greater than or equal to $D = \sqrt{2}\left(\frac{1}{1+S_a}+\frac{1}{1+S_b}+\frac{1}{1+S_c}+\frac{1}{1+S_d}\right)$. We now have $1+S_a = a+ab+abc+abcd = a\cdot(1+S_b) = ab\cdot(1+S_c) = abc(1+S_d)$. Likewise, $1+S_b = bc(1+S_d)$ and $1+S_c = c(1+S_d)$, which yields $D = \sqrt{2}\cdot\frac{1}{1+S_d}\left(\frac{1}{abc}+\frac{1}{bc}+\frac{1}{c}+1\right) = \sqrt{2}$. Thus the statement is proved. ■

Alternate solution:

Choose positive w, x, y, z such that

$$a = \frac{x}{w}, b = \frac{y}{x}, c = \frac{z}{y}, d = \frac{w}{z}$$

Now we can multiply all four new variables by a constant to ensure that $w+x+y+z = \frac{1}{2}$.

Then we have

$$\frac{1}{2}+a+ab+abc = \frac{1}{2}+\frac{x}{w}+\frac{y}{w}+\frac{z}{w} = \frac{1}{2}+\frac{x+y+z}{w} = \frac{1}{2}+\frac{\frac{1}{2}-w}{w} = \frac{1}{2}\cdot\frac{1-w}{w},$$

and similarly for the other three terms. Thus each term of the sum has the form $\sqrt{\frac{2w}{1-w}}$.

Now, to prove a lemma analyzing each of these terms:

$$\text{If } 0 < x < \frac{1}{2}, \text{then } \sqrt{\frac{w}{1-w}} > 2w.$$

Proof:

$$\sqrt{\frac{x}{1-x}} > 2x$$

Square both sides and multiply by $1-x$ to get

$$x > 4x^2\cdot(1-x)$$

Dividing by x and simplifying,

$$0 > 4x\cdot(1-x)-1 = -4x^2+4x-1.$$

This last expression is $-(2x-1)^2$ which is negative since $x < \frac{1}{2}$. (Note that all the steps of this are reversible, so this final true inequality can be used to work backwards and establish our desired inequality.)

Using this result, the original expression is greater than (never equal to)

$$\sqrt{2}\cdot(2w+2x+2y+2x) = \sqrt{2}$$

■