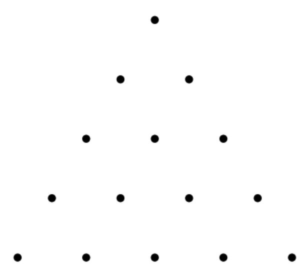
15th Bay Area Mathematical Olympiad

BAMO Exam

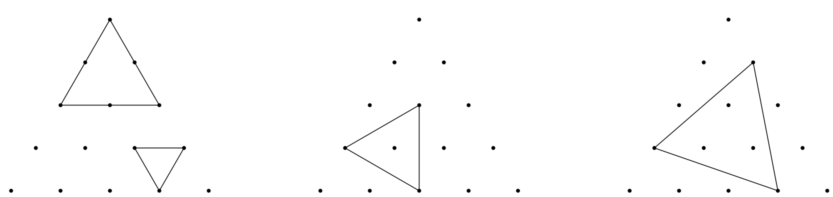
February 26, 2013

Solutions to BAMO-8 and BAMO-12 Problems

1 How many different sets of three points in this equilateral triangular grid are the vertices of an equilateral triangle? Justify your answer.



Solution:



Call the distance between two adjacent dots on the same row 1 unit. Call an equilateral triangle size x if each of its sides are x units long. So the smallest triangle that can be drawn on the grid is size 1, and the largest triangle has size 4.

It is easy to see that in this grid there are only a few possible distances between pairs of points: $1, 2, 3, 4, \sqrt{3}, \sqrt{7}$, $2\sqrt{3}$, and $\sqrt{13}$, so every equilateral triangle that fits in the grid must have its sides equal to one of these lengths.

First, consider triangles with one side horizontal. They can point up or down, and the figure on the left above illustrates a size 1 triangle pointing down and a size 2 triangle pointing up. Each such triangle is completely determined by its uppermost or lowermost point. For upward-pointing triangles there are $4+3+2+1 = 10$ size 1 triangles, $3+2+1 = 6$ size 2 triangles, $2+1 = 3$ size 3 triangles, and 1 size 4 triangle, for a total of 20 upward-pointing triangles. Next, consider triangles with one side horizontal pointing down. There are $3+2+1 = 6$ size 1 triangles and 1 size 2 triangle of this type, for a total of 7 such triangles.

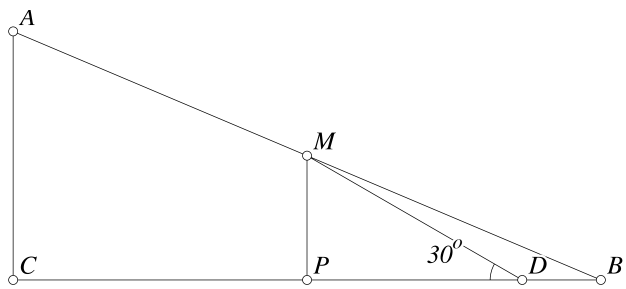
Next, consider tilted triangles that do not have any horizontal sides. There are two possible sizes illustrated by the middle and right-most figures above. The smaller type with side $\sqrt{3}$ have a vertical edge and point either right or left. There are 3 right-pointing triangles a three left-pointing triangles for a total of 6. There are only two of the larger triangles with no horizontal line having side $\sqrt{7}$.

Th only other distances between pairs of points in the grid are $2\sqrt{3}$ and $\sqrt{13}$, and it is easy to see that triangles with these side lengths will not fit.

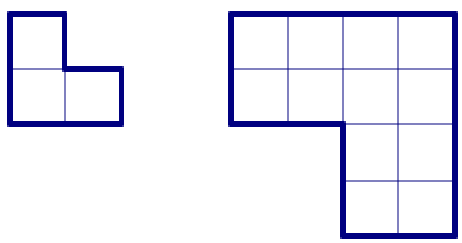
Thus there are $20+7+6+2 = 35$ total triangles. ■

2 Let triangle ABC have a right angle at C, and let M be the midpoint of the hypotenuse AB. Choose a point D on line BC so that angle CDM measures 30 degrees. Prove that the segments AC and MD have equal lengths.

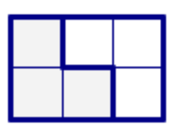
Solution: Drop the perpendicular from M to BC. Let P be the point where this perpendicular meets the line BC. Since $\angle MPB$ and $\angle ACB$ are both right angles, and triangle MPB and triangle ACB share the angle at B, triangle MPB and triangle ACB are similar. Since the length of MB is half the length of BA, side MP must be half the length of AC. But triangle MPD is a $30-60-90$ triangle, so the length of MP is also half the length of MD. Therefore, the length of AC equals the length of MD. ■



3 Define a *size-n tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.



We say that a shape can be *tiled with size-1 trominos* if we can cover the entire area of the shape—*and no excess area*—with *non-overlapping* size-1 trominos. For example, a 2×3 rectangle can be tiled with size-1 trominos as shown below, but a 3×3 square cannot be tiled with size-1 trominos.

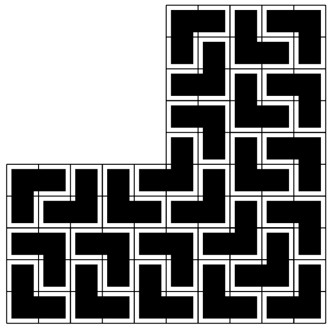


a) Can a size-5 tromino be tiled by size-1 trominos?

b) Can a size-2013 tromino be tiled by size-1 trominos?

Justify your answers.

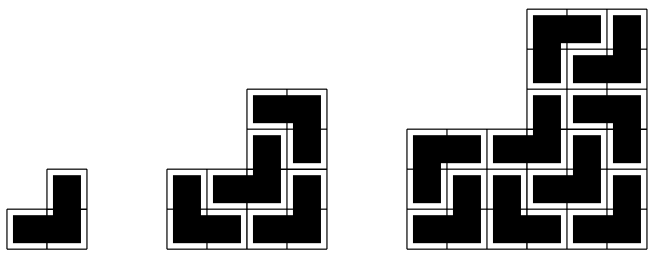
Solution: We will abbreviate "tile with size-1 trominos" with "tile." It is possible to tile a size-5 tromino as drawn.



size 5

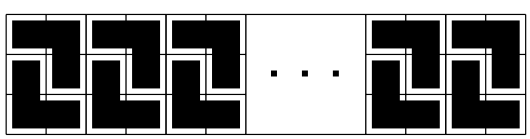
It is also possible to tile a size-2013 tromino. In fact, any size-n tromino can be tiled with size-1 trominos, which can be proved with mathematical induction as follows.

Size-1, size-2, and size-3 trominos can be tiled as shown below.



size 1 size 2 size 3

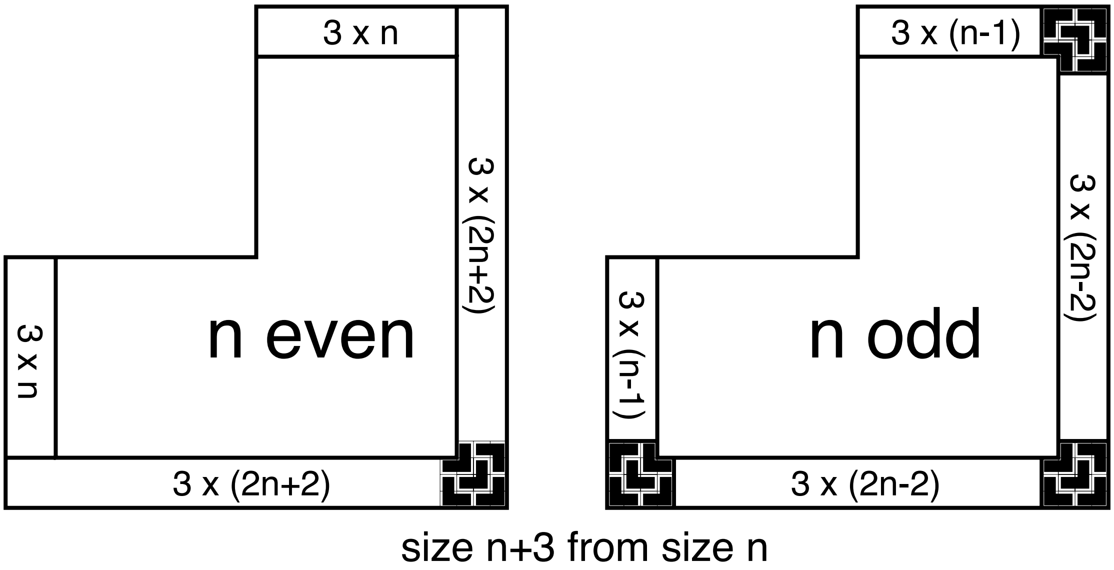
If k is even, it is possible to tile a $3 \times k$ rectangle as shown.



Suppose that a size-n tromino can be tiled. Then we can tile a size-$(n+3)$ tromino as follows.

If n is even, fill in the size-$(n+3)$ tromino with a size-n tromino, plus a border of width 3 that can be made from two $3 \times n$ rectangles at the ends, two $3 \times (2n+2)$ rectangles along the sides, and one corner patch that is a 4×4 square with a corner removed.

If n is odd, fill in the size-$(n+3)$ tromino with a size-n tromino, plus a border that can be made from two $3 \times (n-1)$ rectangles at the ends, two $3 \times (2n-2)$ rectangles at the sides, and three corner patches. See the figure below. ■



4 For a positive integer $n > 2$, consider the $n-1$ fractions

$$\frac{2}{1}, \frac{3}{2}, \cdots, \frac{n}{n-1}.$$

The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. Can you make the product equal 1? Find all values of n for which this is possible and prove that you have found them all.

Solution:

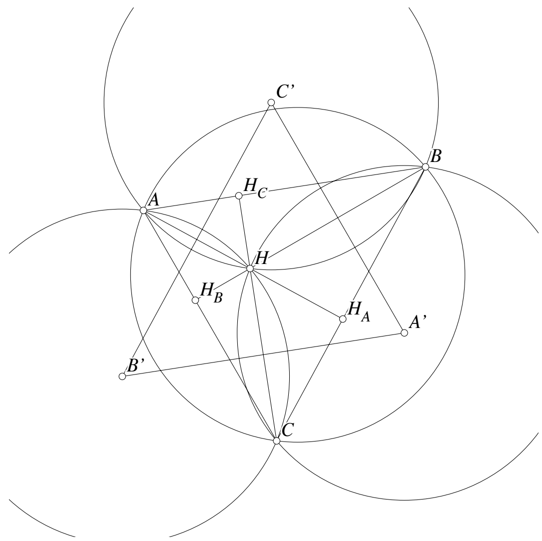
We will show that this is possible exactly when n is a perfect square larger than 1. Suppose that we can reciprocate some of the fractions so that the resulting product is 1. Let r represent the product of the fractions that we will reciprocate and t represent the product of the fractions that we will leave alone. Then $r \cdot t = n$ while $\frac{1}{r} \cdot t = 1$. Multiplying these equations shows that $n = t^2$, so n is the square of a rational number, which means that it has to be a perfect square.

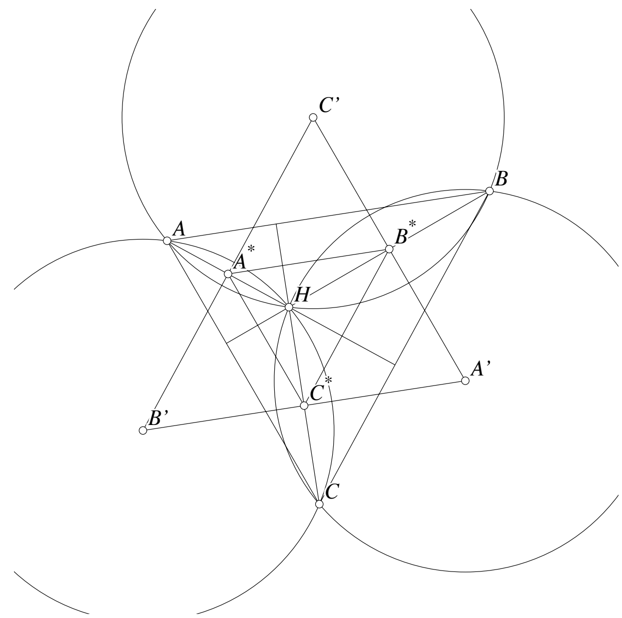
Now suppose that $n = a^2$ is a perfect square. Then we can reciprocate the first $a-1$ terms of the product to obtain

$$\left(\frac{1}{2}\right) \cdots \left(\frac{a-1}{a}\right) \left(\frac{a+1}{a}\right) \cdots \left(\frac{a^2}{a^2-1}\right) = \frac{1}{a} \cdot \frac{a^2}{a} = 1,$$

demonstrating that modifying the product as desired is indeed possible for any perfect square. ∎

5 Let H be the orthocenter of an acute triangle ABC. (The *orthocenter* is the point at the intersection of the three altitudes. An *acute triangle* has all angles less than $90°$.) Draw three circles: one passing through A, B and H, another passing through B, C and H, and finally, one passing through C, A and H. Prove that the triangle whose vertices are the centers of those three circles is congruent to triangle ABC.[1]





Solution 1:

See the figure on the left, above. Since B' and C' are each equidistant from A and H the line $B'C'$ is the perpendicular bisector of AH. The line AH is also an altitude of $\triangle ABC$ so AH is also perpendicular to BC. Since BC and $B'C'$ are both perpendicular to AH they are parallel. Similarly, $AB||A'B'$ and $CA||C'A'$. Therefore we conclude that $\triangle ABC$ is similar to $\triangle A'B'C'$.

The extended law of sines states that for any triangle $\triangle PQR$ inscribed in a circle, the diameter of that circle is equal to $\frac{|PR|}{\sin(\angle PQR)}$.

Therefore, the diameter of the circumcircle of $\triangle BHC$ is $\frac{|BC|}{\sin(\angle BHC)}$ and the diameter of the circumcircle of $\triangle ABC$ is $\frac{|BC|}{\sin(\angle BAC)}$. Since the altitudes of $\triangle ABC$ are perpendicular to the bases, the quadrilateral AH_BHH_C has right angles at H_B and B_C so it is cyclic. Thus $180° - \angle BAC = \angle H_BHH_C$. Because they are vertical angles, we have $\angle H_BHH_C = \angle BHC$, so $180° - \angle BAC = \angle BHC$ and therefore $\sin(\angle BAC) = \sin(\angle BHC)$. Thus the diameters of the two circumcircles mentioned above are equal.

The same argument can be made about the other two circumcircles, so the diameters of all four circles in the diagram on the left above are equal. The point H lies on the three circles centered at A', B' and C' and since all the diameters are equal, H is the circumcenter of $\triangle A'B'C'$ and the diameter of that circumcircle is the same as the diameter of the circumcircle of $\triangle ABC$. Since $\triangle ABC$ and $\triangle A'B'C'$ are similar and have circumcircles with the same diameter, they are congruent. ■

Solution 2:

See the figure on the right, above. Use the same reasoning as in the first solution above to show that $B'C'$, $C'A'$ and $A'B'$ are the perpendicular bisectors of AH, BH and CH, respectively (so $|AA^*| = |A^*H|$, $|BB^*| = |B^*H|$ and $|CC^*| = |C^*H|$). From this it is easy to see that $\triangle A^*B^*C^*$ is homothetic to $\triangle ABC$ with center H and dilation factor $1/2$.

We know that $A^*B^*||AB$ since it is the midsegment of $\triangle AHB$ and similarly $B^*C^*||BC$ and $C^*A^*||CA$. Since by similar reasoning as in the previous solution we know AB, BC and CA are respectively parallel to $A'B'$,$B'C'$ and $C'A'$ we conclude that $A^*B^*||A'B'$, $B^*C^*||B'C'$ and $C^*A^*||C'A'$. The only way this can occur is if $\triangle A^*B^*C^*$ is the

[1] This problem is related to Johnson's Theorem. See, for example, http://en.wikipedia.org/wiki/Johnson_circles

medial triangle of $\triangle A'B'C'$ so they are similar, and $\triangle A^*B^*C^*$ is half the size of the other. It is also similar to and half the size of $\triangle ABC$, so $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

■

6 Consider a rectangular array of single digits $d_{i,j}$ with 10 rows and 7 columns, such that $d_{i+1,j} - d_{i,j}$ is always 1 or -9 for all $1 \le i \le 9$ and all $1 \le j \le 7$, as in the example below. For $1 \le i \le 10$, let m_i be the median of $d_{i,1}, \ldots, d_{i,7}$. Determine the least and greatest possible values of the mean of $m_1, m_2, \ldots, m_{10}$.

Example:

	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$d_{i,4}$	$d_{i,5}$	$d_{i,6}$	$d_{i,7}$	m_i
$i=1$	2	7	5	9	5	8	6	median is 6
$i=2$	3	8	6	0	6	9	7	median is 6
$i=3$	4	9	7	1	7	0	8	median is 7
$i=4$	5	0	8	2	8	1	9	median is 5
$i=5$	6	1	9	3	9	2	0	median is 3
$i=6$	7	2	0	4	0	3	1	median is 2
$i=7$	8	3	1	5	1	4	2	median is 3
$i=8$	9	4	2	6	2	5	3	median is 4
$i=9$	0	5	3	7	3	6	4	median is 4
$i=10$	1	6	4	8	4	7	5	median is 5

Solution 1: Note that rearranging the columns does not change the medians, hence we may sort the first row, so that $d_{1,1} \le d_{1,2} \le \ldots \le d_{1,7}$. The calculations are much simplified if we subtract $i-1$ from each row. In other words, we put $D_{i,j} = d_{i,j} - (i-1)$. This subtracts $i-1$ from the median m_i as well – that is if M_i is the median of $D_{i,j}$s, then $M_i = m_i - (i-1)$. Thus the sum of the M_is is equal to the sum of the m_is minus $0+1+2+\ldots+9 = 45$. We shall show that sum of M_i's is 0, so that the sum of the m_is is 45 and the average is always 4.5.

Note that since $D_{1,1} \le D_{1,2} \le \ldots \le D_{1,7}$ the entry $D_{1,4}$ is a median. The fourth column will continue to contain a median until $d_{i,7} = 0$ (at which point the third column will contain a median), that is $10 - D_{1,7}$ times (note that $d_{1,7} = D_{1,7}$). The sum of those medians is then equal $D_{1,4}(10 - D_{1,7})$. After that, median moves to the third column and stays there until $d_{i,6} = 0$ (this may be no time at all, if $d_{1,6} = d_{1,7}$, but that will not affect the calculation). The contribution of those medians is $D_{1,3}(D_{1,7} - D_{1,6})$. Continuing this way we see that the medians in the second column contribute $D_{1,2}(D_{1,6} - D_{1,5})$ and ones in the first column $D_{1,1}(D_{1,5} - D_{1,4})$. A median then moves to the seventh column, but by that point its value has dropped, $D_{i,7} = D_{1,7} - 10$. The contribution of those medians is then $(D_{1,7} - 10)(D_{1,4} - D_{1,3})$. Similarly for those in sixth and fifth columns we get $(D_{1,6} - 10)(D_{1,3} - D_{1,2})$ and $(D_{1,5} - 10)(D_{1,2} - D_{1,1})$. Finally the median moves to the fourth column again, staying there remaining $D_{1,1}$ times, contributing $(D_{1,4} - 10)D_{1,1}$. Overall, the sum of all medians is thus

$$\begin{aligned} D_{1,4}(10 - D_{1,7}) + D_{1,3}(D_{1,7} - D_{1,6}) + D_{1,2}(D_{1,6} - D_{1,5}) + \\ D_{1,1}(D_{1,5} - D_{1,4}) + (D_{1,7} - 10)(D_{1,4} - D_{1,3}) + (D_{1,6} - 10)(D_{1,3} - D_{1,2}) \\ +(D_{1,5} - 10)(D_{1,2} - D_{1,1}) + (D_{1,4} - 10)D_{1,1}. \end{aligned}$$

It is fairly easy to see that this expression is in fact equal to 0 (for example, by considering the linear and quadratic terms separately). This means that the sum of new medians M_i is zero, and the sum of the original m_i's is 45, as wanted, ■

Solution 2: We will prove a stronger claim: for all a, the number of m_i's equal to a equals the number of m_i's equal to $9-a$. (By a pairing argument, this implies that the average of the m_i's is 9/2.) Indeed, for $1 \le j \le 10$ let $M_{i,j}$ denote the jth smallest entry in row i of the table (so that $m_i = M_{i,4}$); we will show that for all a and j, the number of $M_{i,j}$'s equal to a equals the number of $M_{i,8-j}$'s equal to $9-a$.

Henceforth, all row-indices are to be interpreted modulo 10, and "between" is meant in the inclusive sense.

It follows from the defining property of the table that for all i between 1 and 10, all a between 0 and 9, and all k between 0 and a, the number of k's in row i equals the number of $k+a$'s in row $i+a$. Replacing a by $9-a$, and summing over all k between 0 and a, we find that the number of entries between 0 and a in row i equals the number of entries between $9-a$ and 9 in row $i+9-a$. Hence for all j, the number of entries between 0 and a in row i is greater than or equal to j if and only if the number of entries between $9-a$ and 9 in row $i+9-a$ is greater than or equal to j. But this means that the j smallest entries in row i are all between 0 and a if and only if the j largest entries in row $i+9-a$ are all between $9-a$ and 9. That is, $M_{i,j} \leq a$ if and only if $M_{i+9-a,8-j} \geq 9-a$. Replacing a by $a-1$, we see also that $M_{i,j} \leq a-1$ if and only if $M_{i+10-a,8-j} \geq 10-a$. Combining the last two facts, we conclude that $M_{i,j} = a$ if and only if $M_{i+10-a,8-j} = 9-a$. Summing over i (and noting that $i+10-a$ varies over $0,1,\ldots,9$ mod 10 as i does), we see that the number of i's with $M_{i,j} = a$ equals the number of i's with $M_{i,8-j} = 9-a$, as was claimed above. ■

7 Let $F_1, F_2, F_3 \ldots$ be the *Fibonacci sequence*, the sequence of positive integers with $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. A *Fibonacci number* is by definition a number appearing in this sequence.

Let $P_1, P_2, P_3, \ldots$ be the sequence consisting of all the integers that are products of two Fibonacci numbers (not necessarily distinct), in increasing order. The first few terms are

$$1,2,3,4,5,6,8,9,10,13,\ldots$$

since, for example $3 = 1 \cdot 3, 4 = 2 \cdot 2$, and $10 = 2 \cdot 5$.

Consider the sequence D_n of *successive differences* of the P_n sequence, where $D_n = P_{n+1} - P_n$ for $n \geq 1$. The first few terms of D_n are

$$1,1,1,1,1,2,1,1,3,\ldots.$$

Prove that every number in D_n is a Fibonacci number.

Solution Let $\Phi = \frac{1+\sqrt{5}}{2}$ and $\varphi = \frac{1-\sqrt{5}}{2}$. Note for later use that $\Phi\varphi = -1$, $\Phi - \varphi = \sqrt{5}$, $\Phi = \Phi^2 - 1$, and $\varphi = \varphi^2 - 1$.

We use Binet's formula for the Fibonacci numbers: $F_n = \frac{1}{\sqrt{5}}(\Phi^n - \varphi^n)$. (The reader who is not familiar with this formula may prove it inductively by checking that it works for $n = 1,2$ and is compatible with the Fibonacci recurrence.)

Each P_n may be written as $F_j F_k$ with $j \geq k$. Binet's formula gives

$$\begin{aligned}
F_j F_k &= \frac{1}{5}(\Phi^j - \varphi^j)(\Phi^k - \varphi^k) \\
&= \frac{1}{5}(\Phi^{j+k} + \varphi^{j+k} - \Phi^j \varphi^k - \Phi^k \varphi^j) \\
&= \frac{1}{5}(\Phi^{j+k} + \varphi^{j+k} - (\Phi\varphi)^k(\Phi^{j-k} + \varphi^{j-k})) \\
&= \frac{1}{5}(\Phi^{j+k} + \varphi^{j+k} - (-1)^k(\Phi^{j-k} + \varphi^{j-k})) \\
&= \frac{1}{5}(L_{j+k} - (-1)^k L_{j-k}),
\end{aligned}$$

where we define $L_n = \Phi^n + \varphi^n$. In what follows, we will use two properties of L_n: it is positive for all $n \geq 0$, and $L_{n+4} > L_n$ for all $n \geq 0$. Both properties are easily proved via the observation that L_n is, for all $n \geq 2$, the integer closest to Φ^n.

Now fix r and consider the set of products $F_j F_k$ ($j \geq k$) for which $j + k = r$. All of these products share a "leading" term of $\frac{1}{5}L_r$. The remaining term can be written as $-\frac{(-1)^k}{5}L_{r-2k}$. By the two properties of L_n noted above, we have

$$-L_{r-4} < -L_{r-8} < -L_{r-12} < \cdots < -L_{r-4\lfloor r/4 \rfloor} < L_{r-2-4\lfloor (r-2)/4 \rfloor} < \cdots < L_{r-10} < L_{r-6} < L_{r-2}$$

and thus

$$F_{r-2}F_2 < F_{r-4}F_4 < F_{r-6}F_6 < \cdots < F_{r-5}F_5 < F_{r-3}F_3 < F_{r-1}F_1. \tag{1}$$

We note that the smallest and largest products in inequality (1) are $F_{r-2}F_2 = F_{r-2}$ and $F_{r-1}F_1 = F_{r-1}$, respectively. Thus the largest product $F_j F_k$ with $j+k = r$ is equal to the smallest product $F_j F_k$ with $j+k = r+1$. This implies that the sequence $P_1, P_2, P_3, \ldots$ consists of chains of the form (1) strung end to end for successively increasing values of r. All that remains is to show that the difference between any two consecutive terms in (1) is a Fibonacci number.

Such differences are of the form $\frac{1}{5}(L_{n+2} - L_{n-2})$ (for some integer n), except in the middle where there is one difference of the form $\frac{1}{5}(L_{n+1} + L_{n-1})$. We now show that both of these expressions are equal to F_n:

$$\begin{aligned}
F_n &= \frac{1}{\sqrt{5}}(\Phi^n - \varphi^n) \\
&= \frac{\Phi - \varphi}{5}(\Phi^n - \varphi^n) \\
&= \frac{1}{5}(\Phi^{n+1} + \varphi^{n+1} - \Phi\varphi(\Phi^{n-1} + \varphi^{n-1})) \\
&= \frac{1}{5}(\Phi^{n+1} + \varphi^{n+1} + \Phi^{n-1} + \varphi^{n-1}) \qquad (= \frac{1}{5}(L_{n+1} + L_{n-1})) \\
&= \frac{1}{5}(\Phi^{n+2} - \Phi^n + \varphi^{n+2} - \varphi^n + \Phi^n - \Phi^{n-2} + \varphi^n - \varphi^{n-2}) \\
&= \frac{1}{5}(\Phi^{n+2} + \varphi^{n+2} - \Phi^{n-2} - \varphi^{n-2}) \qquad (= \frac{1}{5}(L_{n+2} - L_{n-2})).
\end{aligned}$$

Therefore, every term of $D_1, D_2, D_3, \ldots$ is a Fibonacci number. ∎