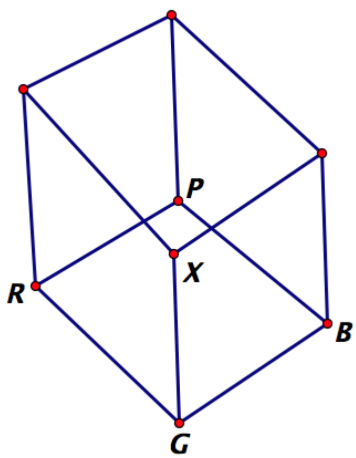
16th Bay Area Mathematical Olympiad

February 25, 2014

BAMO 2014 Problems and Solutions

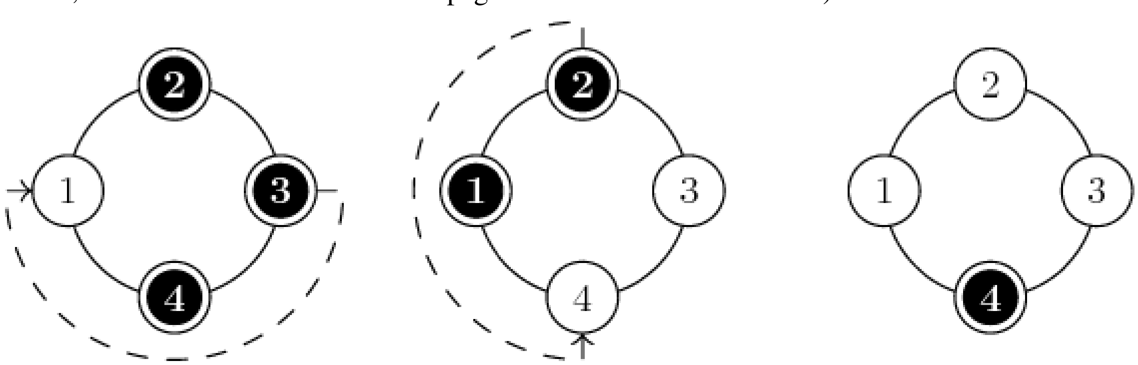
A The four bottom corners of a cube are colored red, green, blue, and purple. How many ways are there to color the top four corners of the cube so that every face has four different colored corners? Prove that your answer is correct.

Solution: There is just one coloring; it's forced: Without loss of generality, suppose that the corners are colored in order, R, G, B, P, as shown below.



Notice that vertex X cannot be R, B, or B; it must therefore be colored P. Likewise, every other vertex on the top has only one possible color, since each top vertex is part of a two faces which collectively use three different colors on the bottom level. So there is only one solution, namely, starting from vertex X and going counterclockwise (as seen from above): P, R, G, B.

B There are n holes in a circle. The holes are numbered $1, 2, 3$ and so on to n. In the beginning, there is a peg in every hole except for hole 1. A peg can jump in either direction over one adjacent peg to an empty hole immediately on the other side. After a peg moves, the peg it jumped over is removed. The puzzle will be solved if all pegs disappear except for one. For example, if $n = 4$ the puzzle can be solved in two jumps: peg 3 jumps peg 4 to hole 1, then peg 2 jumps the peg in 1 to hole 4. (See illustration below, in which black circles indicate pegs and white circles are holes.)



(a) Can the puzzle be solved when $n = 5$?

(b) Can the puzzle be solved when $n = 2014$?

In each part (a) and (b) either describe a sequence of moves to solve the puzzle or explain why it is impossible to solve the puzzle.

Solution: (a) Here is a sketch that shows why there is no solution. After the first move, without loss of generality, there will be two empty locations next to one another, and after the next move, there will be two pegs left, separated by one empty space on one side and by two empty spaces on the other side. After this, it is impossible to proceed, so we are stuck with two pegs. (b) For even n it is pretty easy to construct a solution:

Suppose the holes are numbered 0, 1, 2, ..., $n-1$ with hole 0 initially empty. Then do the following steps: Jump 2 into 0, 4 into 2, 6 into 4, 8 into 6, ... $n-2$ into $n-4$. Now you're left with a peg in $n-1$ next to 0, and holes 2, 4, 6, ..., $n-4$ are filled, so just jump the peg in $n-1$ over all the even pegs like a successive capture in checkers, and you're done.

C and 1 Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won.

Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

Solution: Let's write the number of cookies in the red and blue plate, respectively, as an ordered pair (x, y), so that the legal moves are to $(x-2, y)$ or $(x, y-2)$ or $(x-1, y+1)$. Thus the only positions with no legal move are $(0, 0)$ and $(0, 1)$, and since cookies are eaten in pairs, the final position is determined by the original number of cookies.

Starting from (20,14), we know that eventually all the cookies will be eaten, so there are exactly $(20+14)/2 = 17$ cookie-eating moves. There may also be some number of moves from the first pile to the second pile, but since an even number of cookies are eaten from each pile, there must be an even number of such moves. Thus, the total number of moves in the game is odd, and the first player gets the last legal move.

For general starting positions, there are a few cases to examine depending on whether the total number of cookies is even and the number of cookies in pile 2 is even, but the logic is similar.

D and 2 Let ABC be a scalene triangle with the longest side AC. (A *scalene* triangle has sides of different lengths.) Let P and Q be the points on the side AC such that $AP = AB$ and $CQ = CB$. Thus we have a new triangle BPQ inside triangle ABC. Let k_1 be the circle *circumscribed* around the triangle BPQ (that is, the circle passing through the vertices B, P, and Q of the triangle BPQ); and let k_2 be the circle *inscribed* in triangle ABC (that is, the circle inside triangle ABC that is tangent to the three sides AB, BC, and CA). Prove that the two circles k_1 and k_2 are *concentric*, that is, they have the same center.

Solution: Triangle CBQ is isosceles, so the perpendicular bisector of side BQ is angle bisector of angle C. Similarly for BP and A. The intersection of these two bisectors is the circumcenter of BPQ and the incenter of ABC.

3 Suppose that for two real numbers x and y the following equality is true:

$$(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})=1.$$

Find (with proof) the value of $x+y$.

Solution 1 (Rationalize and Factor): Move the y's to the other side by dividing both sides by $y+\sqrt{1+y^2}$:

$$x+\sqrt{1+x^2}=\frac{1}{y+\sqrt{1+y^2}}.$$

Then get rid of the denominator on the right hand side by multiplying top and bottom of the fraction by $-y+\sqrt{1+y^2}$:

$$x+\sqrt{1+x^2}=\frac{\sqrt{1+y^2)}-y}{(y+\sqrt{1+y^2})(\sqrt{1+y^2}-y)}=\frac{\sqrt{1+y^2)}-y}{(1+y^2)-y^2}=\sqrt{1+y^2}-y.$$

Now move back y to the left and the radical $\sqrt{1+x^2}$ to the right:

$$x+y=\sqrt{1+y^2}-\sqrt{1+x^2}.$$

Now, again "rationalize" the right hand side by multiplying it and dividing by $\sqrt{1+y^2}+\sqrt{1+x^2}$:

$$x+y=\frac{(\sqrt{1+y^2}-\sqrt{1+x^2})(\sqrt{1+y^2}+\sqrt{1+x^2})}{\sqrt{1+y^2}+\sqrt{1+x^2}}=\frac{(1+y^2)-(1+x^2)}{\sqrt{1+y^2}+\sqrt{1+x^2}}=\frac{y^2-x^2}{\sqrt{1+y^2}+\sqrt{1+x^2}}.$$

Move the denominator over the left hand side and factor $y^2-x^2=(y-x)(y+x)$ on the right hand side:

$$(x+y)(\sqrt{1+y^2}+\sqrt{1+x^2})=(x+y)(x-y).$$

Finally, move the right hand side to the left and factor out $(x+y)$:

$$(x+y)(\sqrt{1+y^2}+\sqrt{1+x^2}-(x-y))=0,\ \text{or, equivalently,}\ (x+y)((\sqrt{1+y^2}+y)+(\sqrt{1+x^2}-x))=0.$$

Regardless of what x and y are, we always have $1+y^2>y^2$ and $1+x^2>x^2$, and hence $\sqrt{1+y^2}>|y|$ and $\sqrt{1+x^2}>|x|$, which implies $\sqrt{1+y^2}+y>0$ and $\sqrt{1+x^2}-x>0$. Thus, the second factor in the product above, $(\sqrt{1+y^2}+y)+(\sqrt{1+x^2}-x)$ is always positive. So the product can equal 0 only if $x+y=0$. ■

Solution 2: (Clever Manipulations) Suppose

$$(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})=1.$$

Multiply both sides by $-y+\sqrt{1+y^2}$ to get

$$x+\sqrt{1+x^2}=-y+\sqrt{1+y^2}.$$

Rearrange to get

$$x+y=\sqrt{1+y^2}-\sqrt{1+x^2}. \quad (1)$$

By a symmetrical argument, we also have

$$x+y=\sqrt{1+x^2}-\sqrt{1+y^2}. \quad (2)$$

Average (1) and (2) to conclude that $x+y=0$. ■

Solution 3: (Calculus) We note that $y=-x$ gives $(x+\sqrt{1+x^2})(y+\sqrt{1+y^2}=1+x^2-x^2=1$. Further the function $x+\sqrt{1+x^2}$ is monotone increasing (easy with calculus) so for a given x there is at most one solution to $(x+\sqrt{1+x^2})(y+\sqrt{1+y^2}=1$, hence that solution always is $-x$. ■

Solution 4: (Trigonometry): The equality can't hold if x and y are of the same sign. If $x=0$ then $y=0$ and vice versa. So WLOG, assume $x>0$ $y<0$. There are angles X and Y in $(-\pi/2,\pi/2)$ such that $\cot X=x$ and $\cot Y=y$. Then $\sqrt{1+x^2}=\frac{1}{\sin X}$ and $\sqrt{1+y^2}=-\frac{1}{\sin Y}$, $x+\sqrt{1+x^2}=\frac{\cos X+1}{\sin X}=\cot X/2$, $y+\sqrt{1+y^2}=\frac{\cos y-1}{\sin y}=\tan -Y/2$, so $\tan(-Y/2)\cot(X/2)=1$, so $\tan X/2=\tan -Y/2$, so $X=-Y$. ■

4 Let $F_1,F_2,F_3,\ldots$ be the *Fibonacci sequence*, the sequence of positive integers satisfying

$$F_1=F_2=1 \quad \text{and} \quad F_{n+2}=F_{n+1}+F_n \text{ for all } n\geq 1.$$

Does there exist an $n\geq 1$ for which F_n is divisible by 2014?

Solution: Yes! in fact, $F_{54}=86267571272$ is a multiple of 2014 (although you are not required to find this index), and every 54th Fibonacci number thereafter will be a multiple of 2014. To see why, we write the sequence (mod 2014): our goal is to show that it equals zero eventually. Although conventionally, the Fibonacci sequence starts with $F_1=F_2=1$, we can extend it *backwards*—this is the crux idea—with $F_0=0$.

Next, we can show that the sequence is eventually *periodic*: There are only 2014 different values (mod 2014), and thus 2014^2 possible distinct consecutive pairs of numbers. By the pigeonhole principle, eventually, after at most 2014^2+1 steps, we will see the same consecutive pair repeated, and this will then determine the rest of the sequence, with repeating blocks of the same numbers, *ad infinitum*.

We will be done if the periodic block begins with $F_0=0$, since this would imply infinitely many zeros. But perhaps the periodic block didn't start at the beginning. We will use the "extending backwards" idea to show that, in fact, periodicity must start at the beginning of the sequence (F_0).

Suppose that a periodic block starts at $F_M=a, F_{M+1}=b$, and has length L, in other words, ends at F_{M+L-1}, and suppose that $M>0$. Notice that by going backwards, we can compute $F_{M+L-1}=b-a$, since the next periodic block starts at index $M+L$, and $F_{M+L}=a$ and $F_{M+L+1}=b$. Likewise, we can keep going backwards from F_M to deduce that $F_{M-1}=F_{M+L-1}$ and $F_{M-2}=F_{M+L-2}$, etc., so eventually we will get $F_0=F_L$. So the periodicity starts at F_0 and we are guaranteed to see zeros every L steps.

Remark: This proof shows that the length of the period is at most 2014^2+1, when in fact it was much smaller (namely, 54). It can be proven that for a prime p, the maximum period for divisibility (mod p) is $p+1$.

5 A chess tournament took place between $2n+1$ players. Every player played every other player once, with no draws. In addition, each player had a numerical rating before the tournament began, with no two players having equal ratings.

It turns out there were exactly k games in which the lower-rated player beat the higher-rated player. Prove that there is some player who won no less than $n-\sqrt{2k}$ and no more than $n+\sqrt{2k}$ games.

Solution: We suppose the desired conclusion is false, and seek a contradiction. Refer to the players as $P_0,P_1,\ldots,P_{2n}$ in increasing order of their rating. Call a game an *upset* if the lower-rated player beat the higher-rated player.

Let $r = \lfloor \sqrt{2k} \rfloor$. By assumption, for each value $i = 1, 2, \ldots, r$, player P_{n-i} won either less than $n-r$ or more than $n+r$ games. In the first case, player P_{n-i} must have lost more than $r-i$ games against lower-rated players; in the second case, player P_{n-i} must have won more than $r+i$ games against higher-rated players. Either way, P_{n-i} participated in at least $r-i+1$ upsets.

By similar arguments, for each $i = 1, 2, \ldots, r$, player P_{n+i} also participated in at least $r-i+1$ upsets, and player P_n participated in at least $r+1$ upsets. Thus, altogether, we have at least

$$[r+(r-1)+(r-2)+\cdots+1]+[r+(r-1)+(r-2)+\cdots+1]+(r+1) = (r+1)^2$$

participations in upsets. Every upset involved precisely two players, so this requires a total of at least $(r+1)^2/2$ upsets.

However, $\sqrt{2k} < r+1$, so $k < (r+1)^2/2$. So the number of upsets is less than $(r+1)^2/2$, a contradiction.

Alternative Solution: The following solution, shown in its entirety, received the 2014 Brilliancy Prize for its clarity and beauty, and sense of fun!

Pr 5

Pg 1/3

Brilliancy

7

1978

Call the players $-n, -n+1, -n+2, \ldots, -1, 0, 1, \ldots, n$. Let these be the rankings (so if $m > n$, player m is higher ranked than player n).

Note that when $k=0$, player m wins exactly $n-m$ games. Now suppose $k \geq 1$. Construct the following table with three columns.

	n	
	$\vdots$	
	3	
•	2	
	1	
• •	0	
	-1	• •
	-2	•
	-3	
	$\vdots$	
	$-n$	

For each "upset" of two players $a > b$, place a strawberry on the left of player a in the table and one on the right of player b.

Pb 5

Pg 2/3

1978

Note that exactly $2k$ strawberries are placed. Furthermore, remark that $\forall -n \le m \le n$, player m wins exactly

$$W_m \stackrel{def}{=} m - L_m + R_m \quad \text{games}$$

where L_m, R_m denote the # of strawberries to the left/right of player m.

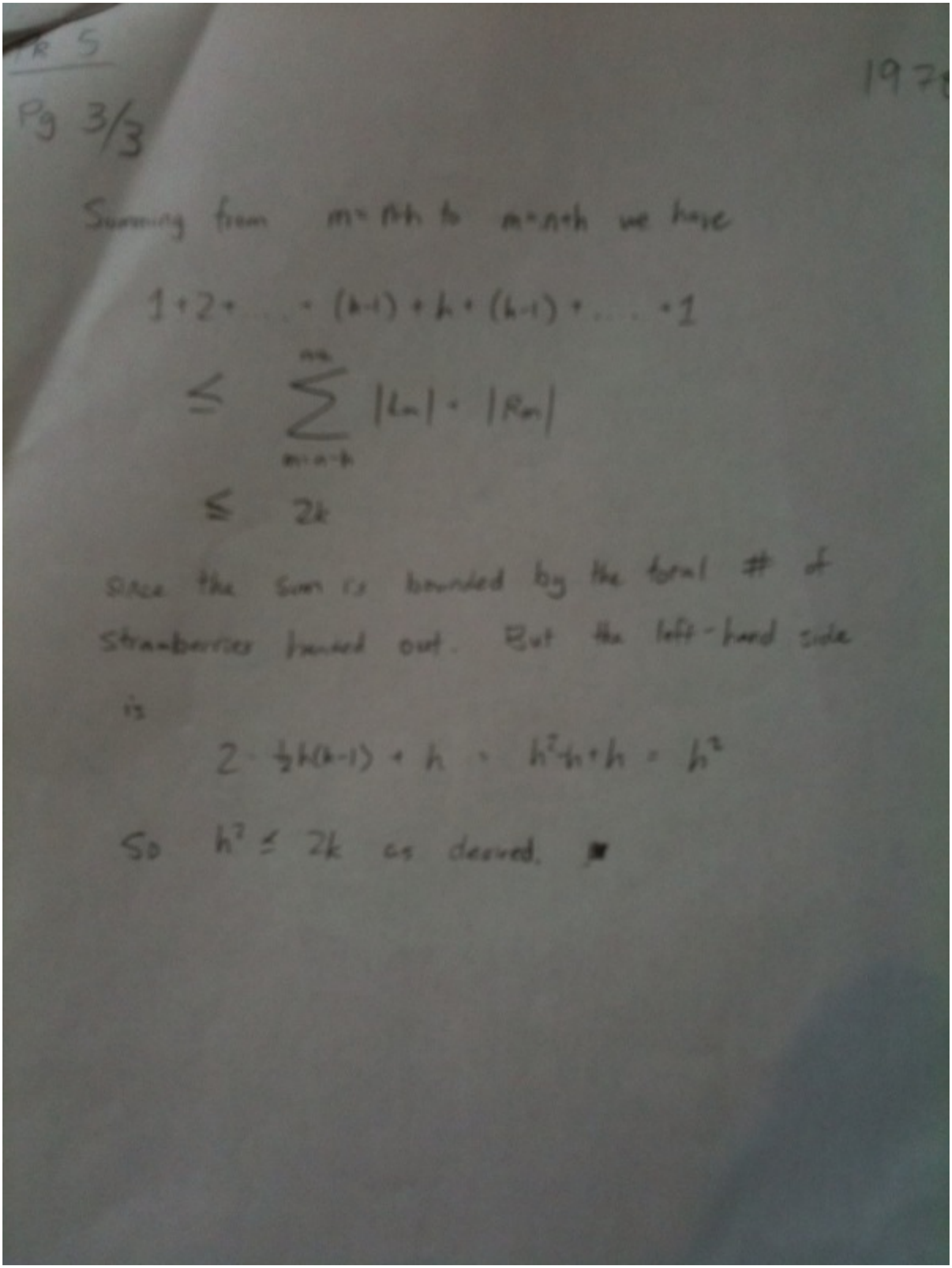
Let

$$h = \min_{-n \le m \le n} |W_m - n| \ge 0.$$

We aim to show $h^2 \le 2k$. Consider a player m. We have

$$\begin{aligned} h &\le |W_m - n| \\ &= |n - m - L_m + R_m| \\ &\le |L_m| + |R_m| + |n - m|. \end{aligned}$$

$$\Rightarrow \quad h - |n - m| \le |L_m| + |R_m|.$$



PR 5

Pg 3/3

1978

Summing from $m = n-h$ to $m = n+h$ we have

$$1+2+\dots+(h-1)+h+(h-1)+\dots+1$$
$$\leq \sum_{m=n-h}^{n+h} |L_m| + |R_m|$$
$$\leq 2k$$

since the sum is bounded by the total # of strawberries handed out. But the left-hand side is

$$2 \cdot \tfrac{1}{2}h(h-1) + h = h^2 - h + h = h^2$$

So $h^2 \leq 2k$ as desired. ■