BAMO

Bay Area Mathematical Olympiad

19th Bay Area Mathematical Olympiad

February 28, 2017

Problems and Solutions

BAMO-8 and BAMO-12 are each 5-question essay-proof exams, for middle- and high-school students, respectively. The problems in each exam are in roughly increasing order of difficulty, labeled A through E in BAMO-8 and 1 through 5 in BAMO-12, and the two exams overlap with three problems. Hence problem C on BAMO-8 is problem #1 on BAMO-12, problem D on BAMO-8 is #2 in BAMO-12, and problem E in BAMO-8 is #4 in BAMO-12.

The solutions below are sometimes just sketches. There are many other alternative solutions. We invite you to think about alternatives and generalizations!

A Consider the 4×4 "multiplication table" below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table. For example, the 3 in the first column times the 4 in the first row give the 12 ($= 3 \cdot 4$) in the cell that is in the 3rd row and 4th column.

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, here is one such possible path, with all the numbers along the path circled:

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

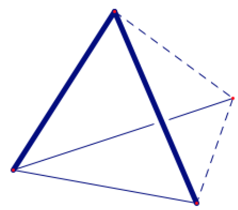
If we add up the circled numbers in the example above (including the start and end squares), we get 48. Considering all such possible paths:

(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.

(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.

B Two three-dimensional objects are said to have the *same coloring* if you can orient one object (by moving or turning it) so that it is indistinguishable from the other. For example, suppose we have two unit cubes sitting on a table, and the faces of one cube are all black except for the top face which is red, and the the faces of the other cube are all black except for the bottom face, which is colored red. Then these two cubes have the same coloring.

In how many different ways can you color the edges of a regular tetrahedron, coloring two edges red, two edges black, and two edges green? (A regular tetrahedron has four faces that are each equilateral triangles. The figure below depicts one coloring of a tetrahedron, using thick, thin, and dashed lines to indicate three colors.)



C/1 Find all natural numbers n such that when we multiply all divisors of n, we will obtain 10^9. Prove that your number(s) n works and that there are no other such numbers.

(*Note:* A *natural* number n is a positive integer; i.e., n is among the counting numbers 1, 2, 3, ... A *divisor* of n is a natural number that divides n without any remainder. For example, 5 is a divisor of 30 because $30 \div 5 = 6$; but 5 is not a divisor of 47 because $47 \div 5 = 9$ with remainder 2. In this problem we consider only positive integer numbers n and positive integer divisors of n. Thus, for example, if we multiply all divisors of 6 we will obtain 36.)

D/2 The area of square $ABCD$ is $196\,\text{cm}^2$. Point E is inside the square, at the same distances from points D and C, and such that $\angle DEC = 150^\circ$. What is the perimeter of $\triangle ABE$ equal to? Prove your answer is correct.

3 Consider the $n \times n$ "multiplication table" below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table.

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, in the case $n = 5$, here is one such possible path, with all the numbers along the path circled:

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

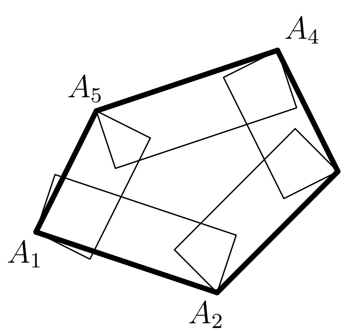
If we add up the circled numbers in the example above (including the start and end squares), we get 93. Considering all such possible paths on the $n \times n$ grid:

(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

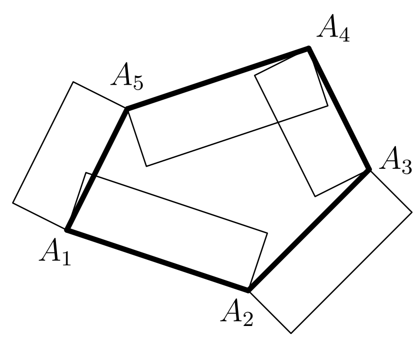
(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

E/4 Consider a convex n-gon $A_1A_2\cdots A_n$. (Note: In a convex polygon, all interior angles are less than $180°$.) Let h be a positive number. Using the sides of the polygon as bases, we draw n rectangles, each of height h, so that each rectangle is either entirely inside the n-gon or partially overlaps the inside of the n-gon.

As an example, the left figure below shows a pentagon with a correct configuration of rectangles, while the right figure shows an incorrect configuration of rectangles (since some of the rectangles do not overlap with the pentagon):



(a) Correct



(b) Incorrect

Prove that it is always possible to choose the number h so that the rectangles completely cover the interior of the n-gon and the total area of the rectangles is no more than twice the area of the n-gon.

5 Call a number T persistent if the following holds: Whenever a,b,c,d are real numbers different from 0 and 1 such that

$$a+b+c+d=T$$

and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=T \ ,$$

we also have

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\frac{1}{1-d}=T \ .$$

(a) If T is persistent, prove that T must be equal to 2.

(b) Prove that 2 is persistent.

Note: alternatively, we can just ask "Show that there exists a unique persistent number, and determine its value".

Solutions

A The minimum is 46 and the maximum is 50. To see this more easily, tilt the grid 45 degrees:

```
         1
      2     2
   3     4     3
4     6     6     4
   8     9     8
     12    12
        16
```

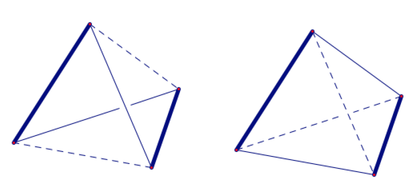
Now every path must include exactly one number from each row. The smallest and largest numbers in each row are respectively at the edge and in the middle, so the smallest and largest totals are achieved by the paths below:

```
         1                            1
      2     2                      2     2
   3     4     3                3     4     3
4     6     6     4          4     6     6     4
   8     9     8                8     9     8
     12    12                     12    12
        16                           16
```

These totals are 46 and 50.

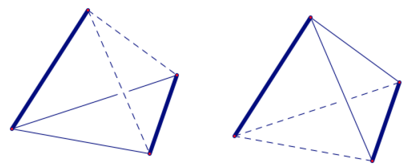
B There are nine non-equivalent colorings where two edges are red, two are black, and two are green. In the illustrations below, we will indicate the three colors by drawing edges thick, thin, or dashed. We will partition the colorings into three cases, determined by the number of pairs of edges that are colored the same: three, one, or zero (two is impossible). For each case, without loss of generality, we fix the position of the two thick edges and carefully examine all possible colorings, and determine which are equivalent.

- *Each pair of opposite edges is colored the same.* Fixing the two thick edges, there are two possible configurations shown below.



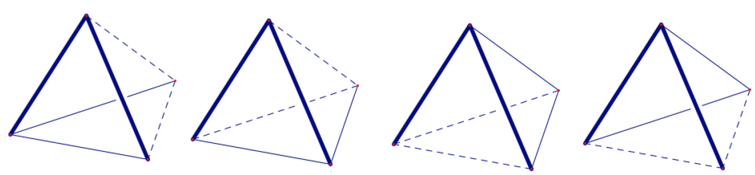
These two colorings are non-equivalent. To see why, note that the only rotations available are 2-fold (180-degree) rotations about axes joining midpoints of opposite sides, or 3-fold (120-degree) rotations about axes joining a vertex with the center of the opposite face. The 2-fold rotations leave both pictures alone (none of them turns one picture into the other), and the 3-fold rotations do not keep the thick lines in place.

- *Exactly one pair of opposite sides is the same color.* For example, suppose that the two opposite sides are colored with thick lines. Keeping this pair in place, there are only two configurations.



Note that these two colorings are in fact equivalent, since you can turn one into the other by performing a 2-fold rotation about the axis joining the midpoint of the two thick edges. Since there are three choices of colors we can use for the pair, this case has a total of 3 non-equivalent colorings.

- *No pairs of opposite edges have the same color.* Again, we will fix two thick edges and carefully count the possibilities.



These are the only possible choices, once we fix the thick edges, and they are all non-equivalent, since no rotation will keep the two thick edges in place. Furthermore, the first two tetrahedra each have a thick-thick-thin face, but the first one has a face with edges colored thick-thin-dashed (going clockwise), whereas the second has a thick-thin-dashed face (going *counterclockwise*). For the next two tetrahedra, both have thick-thick-dashed faces, but the first has a thick-dashed-thin clockwise face, but the last has a thick-dashed-thin counterclockwise face.

In sum, there are $2+3+4=9$ non-equivalent colorings.

Brilliancy Prize Alternative Solution: Suraj Mathashery of Kennedy Middle School wrote a nice solution that was the only paper which correctly applied the "count all the colorings as a string and divide out by the symmetries" method. Many people correctly counted (exercise) that there are 90 ways to arrange the letters RRGGBB as a string, and then divided this number "because of rotations" to get an answer. The problem is that there is no single number one can choose! Here is a sketch of Suraj's method. You are urged to fill in the gaps.

- There are 90 colorings if we don't worry about any rotations.
- These fall into two categories: The colorings where all three pairs of opposite edges are colored the same way, and those colorings where this is not the case.

- The first category has just 6 colorings; the second has $90 - 6 = 84$.
- The first category has just three rotations that change it; while the second category has 12.
- Hence the number of non-equivalent colorings is $6/3 + 84/12 = 2 + 7 = 9$.

C/1 Solution 1: Since the prime factorizaton of $10^9 = 2^9 \cdot 5^9$, the prime divisors of our natural number n are exactly 2 and 5; i.e., $n = 2^a \cdot 5^b$ for some integer exponents $a \geq 1$ and $b \geq 1$.

When $a = b = 2$, we get the number $n = 2^2 \cdot 5^2 = 100$, which actually works: the product of its divisors is:

$$\begin{aligned} 1 \cdot 2 \cdot 4 \cdot 5 \cdot 10 \cdot 20 \cdot 25 \cdot 50 \cdot 100 &= (1 \cdot 100) \cdot (2 \cdot 50) \cdot (4 \cdot 25) \cdot 10 \\ &= 100^4 \cdot 10 = 10^2 \cdot 10^2 \cdot 10^2 \cdot 10^2 \cdot 10 = 10^9. \end{aligned}$$

In fact, $n = 100$ is the only number that works. Indeed, if in $n = 2^a \cdot 5^b$ we have, say, $b \geq 3$, we will end up with too many 5s in the product of all divisors! To see this, note that we will have at least the divisors 5, 25, 125, $2 \cdot 5$, $2 \cdot 25$, and $2 \cdot 125$. This yields at least $1 + 2 + 3 + 1 + 2 + 3 = 12$ fives in 10^9, but we have only 9 such 5s, a contradiction. Similarly, in $a \geq 3$, we will end up with at least 12 twos in 10^9, which is also a contraduction. We conclude that $a \leq 2$ and $b \leq 2$.

On the other hand, if, say $b = 1$, we will get too few fives in the product! Indeed, $n = 2^a \cdot 5$ with $a \leq 2$, so that the 5s in the product will participate in at most the following divisors: $5, 2 \cdot 5, 2^2 \cdot 5$; i.e., only at most 3 fives, but we need 9 fives in 10^9, a contradiction! Similarly, we will get too few 2s in the product if $a = 1$. We conclude that $a = b = 2$ and the only number that works is $n = 2^2 \cdot 5^2 = 100$. ■

Solution 2: If we arrange all divisors of n in increasing order, $d_1 < d_2 < d_3 < \ldots < d_k$, and multiply the first and the last divisor, the second and the last but one divisor, etc., we will always obtain n (why?). This means that if we multiply the product $\Pi = d_1 \cdot d_2 \cdot d_3 \cdots d_k$ by itself, we will obtain:

$$\Pi^2 = (d_1 d_k)(d_2 d_{k-1}) \cdots (d_k d_1) = \underbrace{n \cdot n \cdots n}_{k} = n^k,$$

where k is the number of divisors of n.

We are given that the product $\Pi = 10^9$, so we conclude that $(10^9)^2 = n^k$, i.e., $n^k = 10^{18}$. From here it is easy to see that n must be a power of 10; i.e., $n = 10^c$ for some natural c such that $(10^c)^k = 10^{18}$. Thus, $10^{ck} = 10^{18}$, $ck = 18$, and c is a divisor of 18. One may think that we need now check all cases for c, but there are faster ways to proceed. You may want to check that $c = 2$ works ($n = 100$ works as in Solution 1), argue that 10^3 and higher powers of 10 will yield larger than 10^9 products Π and that lower powers of 10 (i.e., 10^1) will yield smaller than 10^9 products Π.

Alternatively, note that $n = 10^c = 2^c \cdot 5^c$ means that n will have $k = (c+1) \cdot (c+1)$ divisors of n (why?), and hence $ck = 18$ becomes $c(c+1)(c+1) = 18$, which has only one natural solution of $c = 2$ (why?).

Summarizing, $n = 10^c = 10^2 = 100$ is the only possible solution. ■

D/2 Solution 1: Since the area of square $ABCD$ is $196 = 14^2\,(\text{cm}^2)$, then the side of square $ABCD$ is 14 cm.

We claim that $\triangle ABE$ is equilateral. To prove this, we make a reverse construction, starting from an equilateral $\triangle ABE'$, building up to square $ABCD$, and eventually showing that points E and E' coincide. Thus,

Step 1. Let E' be a point inside square $ABCD$ such that $\triangle ABE'$ is equilateral.
Hence, in particular, $\angle ABE' = 60^\circ$.

Step 2. Then $\angle E'BC = 90^\circ - 60^\circ = 30^\circ$.

Step 3. Since $BE' = AB = BC$ ($\triangle ABE'$ is equilateral and $ABCD$ is a square), we conclude that $\triangle CE'B$ is isosceles with $BE' = BC$ and $\angle E'BC = 30^\circ$.

Step 4. This in turn implies that both base angles of $\triangle CE'B$ are $\frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$. In particular, $\angle BCE' = 75^\circ$. (*Note:* Since $\angle BCE' < 90^\circ$, point E' is between parallel lines AB and CD, and hence E' is indeed *inside* square $ABCD$, and **not** "above" CD or outside $ABCD$.)

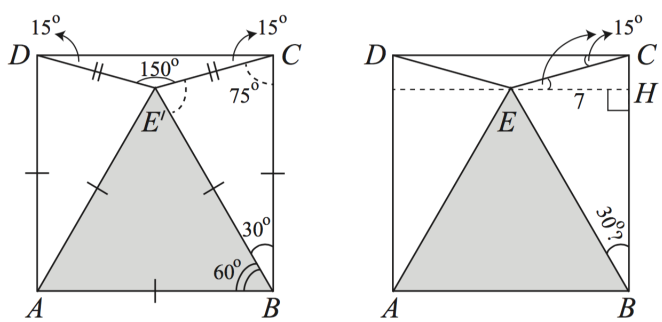
Step 5. But then $\angle DCE' = 90^\circ - 75^\circ = 15^\circ$.

Step 6. Analogously (or by symmetry) we can show that $\angle CDE' = 15^\circ$.

Step 7. This means that $\triangle DCE'$ is isosceles with $DE' = CE'$ and base $\angle DE'C = 150^\circ$.

Step 8. Thus, E' is at the same distances from C and D, $\angle DE'C = 150^\circ$, and E' is inside square $ABCD$. But there is only one such point with all these properties, namely, the given point E. We conclude that point E' is, after all, the same as point E. ■

To finish off the problem, we use the fact that $\triangle ABE'$ is equilateral by construction; i.e., the original $\triangle ABE$ is equilateral. Hence its perimeter is equal to $3 \cdot AB = 3 \cdot 14 = 42\,\text{cm}$. ■



Problem D: Solution 1 Problem D: Solution 2

Solution 2: Our goal will be to confirm that $\angle EBC = 30^\circ$. For those who know some trigonometry, recall a formula for tan(half-angle):

$$\tan^2\alpha = \frac{\cos^2\alpha}{\sin^2\alpha} = \frac{\frac{1}{2}(1-\cos 2\alpha)}{\frac{1}{2}(1+\cos 2\alpha)} \Rightarrow \tan^2 15^\circ = \frac{1-\cos 30^\circ}{1+\cos 30^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{2-\sqrt{3}}{2+\sqrt{3}}.$$

With the help of a little bit of algebra, we rationalize the denominator and obtain:

$$\frac{(2-\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{(2-\sqrt{3})^2}{2^2-3} = \frac{(2-\sqrt{3})^2}{1} \Rightarrow \tan 15^\circ = +\sqrt{(2-\sqrt{3})^2} = 2-\sqrt{3} \ (>0).$$

Now back to our problem. Drop a perpendicular from point E to side BC of the square and mark by H the foot of this perpendicular. Since E is at the same distances from C and D, it easily follows that E is half-way between parallel lines AD and BC; i.e., $EH = \frac{1}{2}AB = 7\,\text{cm}$.

Just as above, from isosceles $\triangle DEC$ we know that $\angle ECD = 15^\circ$. This means that $\angle CEH = 15^\circ$ ($EH||DC$, CE is a transversal, and $\angle ECD$ and $\angle CEH$ are alternating interior angles). Thus, from right $\triangle EHC$ we can calculate $CH = 7\tan 15^\circ$. This implies that $BH = BC - CH = 14 - 7\tan 15^\circ$.

We are ready to calculate $\cot\angle EBC$ from right $\triangle EHB$:

$$\cot\angle EBC = \cot\angle EBH = \frac{BH}{EH} = \frac{14 - 7\tan 15^\circ}{7} = 2 - \tan 15^\circ = 2 - (2 - \sqrt{3}) = \sqrt{3}.$$

But $\cot 30^\circ = \sqrt{3}$, so the acute $\angle EBC$ must be 30°. We proved what we aimed for! ■

To finish off the problem, note that $\angle ABE = 90^\circ - \angle EBC = 90^\circ - 30^\circ = 60^\circ$. By symmetry, $\angle ABE = 60^\circ$ and $\triangle ABE$ is equilateral. Thus, its perimeter is $3AB = 42\,\text{cm}$. ■

Solution 3: It is not hard to compute the ratios of a 15-75-90 right triangle without any trig, provided one knows the ratios of the 30-60-90 triangle. Just divide the 75-degree angle into 15-degree and 60-degree angles, dissecting the original triangle into a 15-15-150 and 30-60-90. Quite elementary from there. ■

3 The minimum is achieved by staying on the perimeter of the grid, and the maximum by staying as close to the main diagonal as possible. To see why this is true, tilt the grid 45 degrees:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 2 & & \\ & 3 & & 4 & & 3 & \\ 4 & & 6 & & 6 & & 4 \\ \iddots & & & \vdots & & & \ddots \end{array}$$

Now every path must include exactly one number from each row. The n^{th} row consists of the numbers $k(n+1-k)$ for $k = 1, 2, \ldots, n$. As a quadratic function of k, this expression is greatest when $k = (n+1)/2$, and gets smaller as k gets farther from $(n+1)/2$ in either direction. Thus the smallest number in row n of our tilted grid is the number on either end of the row, which is n, and the largest number in row n is the number closest to the center, which is $\lfloor \frac{n+1}{2} \rfloor \cdot \lceil \frac{n+1}{2} \rceil$. By staying on the perimeter of the grid, we may select the smallest entry from every row, and so accumulate the smallest possible total. By staying in the middle, we may select the largest entry from every row and accumulate the largest possible total.

All that remains is to calculate these totals in terms of n.

The minimum total is

$$\begin{aligned}(1+2+3+\cdots+(n-1)) + (n+2n+3n+\cdots+n^2) &= (1+2+\cdots+(n-1)) + n(1+2+\cdots+n) \\ &= \frac{n(n-1)}{2} + n\cdot\frac{n(n+1)}{2} \\ &= \frac{n(n^2+2n-1)}{2}.\end{aligned}$$

The maximum total is

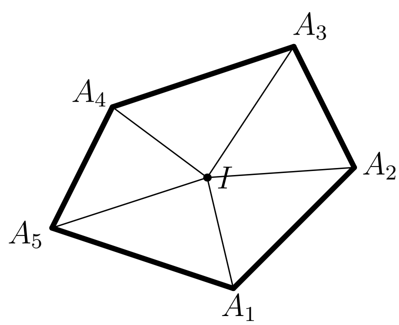
$$\begin{aligned}
1^2+1\cdot 2+2^2+2\cdot 3+\cdots+(n-1)\cdot n+n^2 &= (1^2+2^2+\cdots+n^2)+(1\cdot 2+2\cdot 3+\cdots+(n-1)\cdot n)\\
&= \frac{n(n+1)(2n+1)}{6}+2\cdot\left[\binom{2}{2}+\binom{3}{2}+\cdots+\binom{n}{2}\right]\\
&= \frac{n(n+1)(2n+1)}{6}+2\binom{n+1}{3}\\
&= \frac{n(n+1)(2n+1)}{6}+\frac{2(n-1)(n)(n+1)}{6}\\
&= \frac{n(n+1)(4n-1)}{6},
\end{aligned}$$

and we are finished.

4 For every point P in the interior of Π, let $f(P)$ be the minimum distance from P to a point on the perimeter of Π. Let r denote the maximum value of $f(P)$, which necessarily exists, and let this maximum be achieved at some (not necessarily unique) point $P=I$. (We remark that in the case where Π is a triangle, I is the incenter.)

We claim that if we choose $t=r$, then the rectangles described in the problem meet the desired requirements. First, we show that the rectangles cover Π. Let Q be any point in the interior of Π, and let R be any closest point to Q on the perimeter of Π; observe that $QR=f(Q)\le r$. Let R be on side A_iA_{i+1} of Π. Then we show by contradiction that segment QR is perpendicular to A_iA_{i+1}. For if this is not the case, then let R' be the foot of a perpendicular from Q to the line A_iA_{i+1}. We have $QR'\le QR$, so either $R'=R$ or R' lies outside Π on an extension of side A_iA_{i+1}. But if R' lies outside Π, then the segment QR' must cross the perimeter of Π at some point R'', in which case $QR''<QR'\le QR$, contradicting our choice of R. Thus we have shown that QR is perpendicular to A_iA_{i+1}, and that $QR\le r$. It follows that Q lies in or on the rectangle $A_iA_{i+1}B_iC_i$. Hence all points in the interior of Π are covered by the rectangles.

Finally, we show that the total area of the rectangles is at most twice the area of Π. To see this, draw the segments $IA_1, IA_2, \ldots, IA_n$, dissecting Π into n triangles:



Denoting the area of a polygon by brackets, we have

$$[\Pi]=[A_1IA_2]+[A_2IA_3]+\cdots+[A_nIA_1].$$

Let the i^{th} triangle ($1\le i\le n$) have base $b_i=A_iA_{i+1}$ and corresponding altitude h_i, which is the perpendicular distance from I to line A_iA_{i+1}. The foot of the altitude is either on the perimeter of Π or outside Π, so we have $h_i\ge f(I)=r$. Hence

$$\begin{aligned}
[\Pi] &= \frac{1}{2}b_1h_1+\frac{1}{2}b_2h_2+\cdots+\frac{1}{2}b_nh_n\\
&\ge \frac{1}{2}(b_1+b_2+\cdots+b_n)(r).
\end{aligned}$$

The total area of the rectangles $A_iA_{i+1}B_iC_i$ is $(b_1+b_2+\cdots+b_n)(r)$. The above work shows that this is less than or equal to $2[\Pi]$, so we are finished.

5 (a) Suppose T is persistent. Observe that for any u satisfying $|u|\ge 2$, the equation $x+1/x=u$ has real solutions (because it can be written in the form $x^2-ux+1=0$, which has discriminant u^2-4). Choose u large enough so that both u and $v=T-u$ have absolute value greater than 2. Then there exist real x,y, not equal to 0 or 1, such that $x+1/x=u$ and $y+1/y=v$. Let $a=x$, $b=1/x$, $c=y$, $d=1/y$. Then $a+b+c+d=1/a+1/b+1/c+1/d=u+v=T$, while

$$\begin{aligned}
T &= \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\frac{1}{1-d}\\
&= \frac{1}{1-x}+\frac{1}{1-1/x}+\frac{1}{1-y}+\frac{1}{1-1/y}\\
&= \frac{1}{1-x}+\frac{x}{x-1}+\frac{1}{1-y}+\frac{y}{y-1}\\
&= \frac{1-x}{1-x}+\frac{1-y}{1-y}\\
&= 2,
\end{aligned}$$

and we are finished.

(b) Now we will show that 2 is persistent.

Suppose $a+b+c+d=2$ and $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=2$. Consider the monic polynomial $f(x)$ with roots a,b,c,d. Let $f(x)$ have expansion

$$f(x)=x^4-e_1x^3+e_2x^2-e_3x+e_4.$$

By Vieta's formulas, we have

$$e_1=a+b+c+d=2$$

and

$$\frac{e_3}{e_4}=\frac{bcd+cda+dab+abc}{abcd}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=2.$$

Thus, $f(x)$ has the form $x^4-2x^3+rx^2-2sx+s$.

Next, we consider the polynomial $g(x)=f(1-x)$, which is the monic polynomial with roots $1-a$, $1-b$, $1-c$, $1-d$. We have

$$\begin{aligned}
g(x) &= (1-x)^4-2(1-x)^3+r(1-x)^2-2s(1-x)+s\\
&= x^4-2x^3+rx^2+(2-2r+2s)x-(1-r+s).
\end{aligned}$$

By Vieta's formulas again, we have

$$\begin{aligned}
\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}+\frac{1}{1-d} &= \frac{-(2-2r+2s)}{-(1-r+s)}\\
&= 2.
\end{aligned}$$

Therefore, 2 is persistent. ■