21st Bay Area Mathematical Olympiad

Problems and Solutions

February 26, 2019

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

Problem A

Let a and b be positive whole numbers such that $\frac{4.5}{11} < \frac{a}{b} < \frac{5}{11}$. Find the fraction $\frac{a}{b}$ for which the sum $a+b$ is as small as possible. Justify your answer.

Solution

By multiplying numerators and denominators by 7, we can rewrite the inequalities as follows:

$$\frac{7 \cdot 4.5}{7 \cdot 11} < \frac{a}{b} < \frac{7 \cdot 5}{7 \cdot 11} \Rightarrow \frac{31.5}{77} < \frac{a}{b} < \frac{35}{77}.$$

We now see that the fraction $\frac{a}{b} = \frac{33}{77}$ works: $\frac{31.5}{77} < \frac{33}{77} < \frac{35}{77}$. The fraction reduces to $\frac{a}{b} = \frac{3}{7}$, and its sum $a+b$ is $3+7=10$.

But is this the minimal possible sum $a+b$? To show that no other fraction with a smaller a works, we multiply the inequalities by $11b$ and solve for b:

$$4.5b < 11a < 5b \Rightarrow \frac{11a}{5} < b < \frac{11a}{4.5}.$$

Now, if $a=1$ we obtain $\frac{11}{5} < b < \frac{11}{4.5}$. However, $2 < \frac{11}{5}$ and $\frac{11}{4.5} < 3$ (why?), so b is now squeezed between 2 and 3: $2 < b < 3$, which is nonsense since b is a whole number. We analogously eliminate the case when $a=2$ by arriving at an impossible situation for the whole number b:

$$\frac{11 \cdot 2}{5} < b < \frac{11 \cdot 2}{4.5} \Rightarrow 4 < \frac{22}{5} < b < \frac{22}{4.5} < 5 \Rightarrow 4 < b < 5, \text{ a contradiction!}$$

Thus, $a \neq 1$ and $a \neq 2$, but $a=3$ is possible, with $b=7$. Moreover,

$$\frac{a}{b} < \frac{5}{11} < \frac{5}{10} = \frac{1}{2} \Rightarrow 2a < b.$$

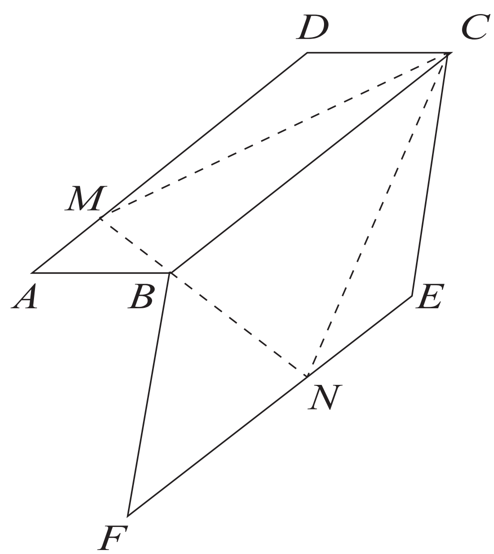
Thus, when $a=3$, $b>6$, i.e., $b=7$ is the smallest possible denominator for $a=3$, yielding the sum $a+b=3+7=10$.

Finally, if $a \geq 4$, $a+b > a+2a = 3a \geq 3 \cdot 4 = 12 > 10$. Thus, a cannot be 4 or more and yield the smallest possible sum.

We conclude that the fraction $\frac{3}{7}$ fits the two inequalities and has the smallest sum of numerator and denominator. ■

Problem B

In the figure below, parallelograms $ABCD$ and $BFEC$ have areas 1234 cm^2 and 2804 cm^2, respectively. Points M and N are chosen on sides AD and FE, respectively, so that segment MN passes through B. Find the area of $\triangle MNC$.



Solution

The area of parallelogram $ABCD$ is BC times the perpendicular distance from BC to AD. The area of triangle $\triangle BCM$ is half the base BC times the perpendicular distance from BC to AD. Therefore the area of $\triangle BCM$ is half the area of parallelogram $ABCD$; that is, $1234/2 = 617$ cm^2.

Similarly, the area of $\triangle BCN$ is half the area of parallelogram $BFEC$; that is, $2804/2 = 1402$ cm^2. Since the area of $\triangle MNC$ is the sum of the two triangular areas, we obtain that its area is $617 + 1402 = 2019$ cm^2.

Note that the answer is the average of the areas of the two parallelograms. ■

Problem C/1

You are traveling in a foreign country whose currency consists of five different-looking kinds of coins. You have several of each coin in your pocket. You remember that the coins are worth 1, 2, 5, 10, and 20 florins, but you have no idea which coin is which and you don't speak the local language. You find a vending machine where a single candy can be bought for 1 florin: you insert any kind of coin, and receive

1 candy plus any change owed. You can only buy one candy at a time, but you can buy as many as you want, one after the other.

What is the least number of candies that you must buy to ensure that you can determine the values of all the coins? Prove that your answer is correct.

Solution

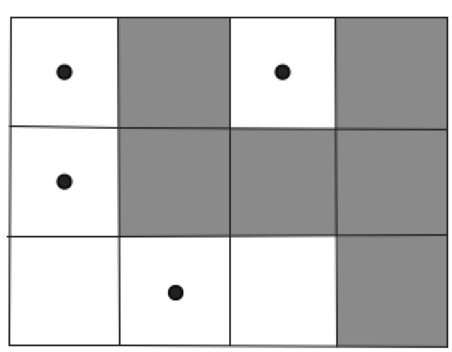
The answer is four.

First we show that three candies are not always enough. If you only buy three candies, then it is possible the three coins you spend will be some combination of 1-, 2-, and 5-florin coins, in which case you definitely won't receive any 10- or 20-florin coins in change. Thus, in this situation, you do not get any information that can be used to distinguish the 10- and 20-florin coins.

Now we show that four candies are enough. In each of these transactions, pay with a different kind of coin. If any of the transactions does not produce change, then you must have paid in that transaction with a 1-florin coin. If all the transactions produce change, then the coin you didn't use as payment is the 1-florin coin. In either case, you can identify the 1-florin coin. Since the change for 2 florins will be 1 florin, you can now tell if you paid for any of the transactions with a 2-florin coin, and thereby identify which of the five coins has a value of 2 florins. Next, we know that the change for 5 florins will be either two 2-florin coins or four 1-florin coins, or one 2-florin and two 1-florin coins; in any event, you can recognize these collections of coins and then deduce which coin is the 5-florin coin (if you see one of these collections, then you know which coin was the 5-florin coin; otherwise, you know it was the coin you didn't use as payment). Likewise, change for 10 florins will be a collection of coins that total 9 florins, and all such collections can now be recognized. Thus we can deduce which coin is the 10-florin coin, and finally we can apply the same procedure to deduce the identity of the 20-florin coin.

Problem D/2

Initially, all the squares of an 8×8 grid are white. You start by choosing one of the squares and coloring it gray. After that, you may color additional squares gray one at a time, but you may only color a square gray if it has exactly 1 or 3 gray neighbors at that moment (where a neighbor is a square sharing an edge). For example, the configuration below (of a smaller 3×4 grid) shows a situation where six squares have been colored gray so far. The squares that can be colored at the next step are marked with a dot.



Is it possible to color all the squares gray? Justify your answer.

Solution

It is not possible. Let $L(t)$ denote the length of the boundary of the gray region after t squares have been colored gray. We have $L(1) = 4$ since the perimeter of the first square colored is 4. After $t \geq 1$ squares have been colored, if we add a square that has exactly one gray neighbor, the edge shared with that neighbor disappears from the boundary but is replaced by three new edges, so $L(t+1) = L(t) + 2$. If we add a square that has exactly three neighbors, all three neighboring edges disappear from the boundary, but there will be one new edge, so $L(t+1) = L(t) - 2$. In either case, every move after the first changes L by 2 (mod 4).

Since $L(1)$ is a multiple of 4, $L(t)$ will be a multiple of 4 whenever t is odd, and $L(t)$ will be congruent to 2 (mod 4) whenever t is even ($t \geq 2$). Thus if all 64 squares could be colored in, $L(64)$ would be congruent to 2 (mod 4). But $L(64)$ would have to equal the perimeter of the grid, which is 32, a multiple of 4. This is a contradiction.

Problem E/3

In triangle $\triangle ABC$, we have marked points A_1 on side BC, B_1 on side AC, and C_1 on side AB so that AA_1 is an altitude, BB_1 is a median, and CC_1 is an angle bisector. It is known that $\triangle A_1B_1C_1$ is equilateral. Prove that $\triangle ABC$ is equilateral too.

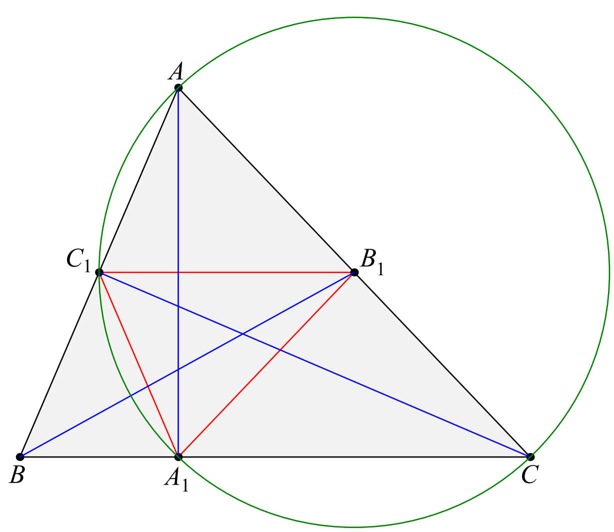
(*Note:* A median connects a vertex of a triangle with the midpoint of the opposite side. Thus, for median BB_1 we know that B_1 is the midpoint of side AC in $\triangle ABC$.)

Solution

Let equilateral $\triangle A_1B_1C_1$ have sides of length s. Since $\triangle AA_1C$ is right with midpoint B_1 on the hypotenuse AC,

$$B_1A = B_1C = B_1A_1 = s = B_1C_1,$$

so points A, C_1, A_1, and C lie on a circle centered at B_1. Therefore, $\triangle ACC_1$ is also right with hypotenuse AC. In other words CC_1 is an altitude, but since it was an angle bisector, we conclude $AC = BC$.



In particular, C_1 is the midpoint of AB. But since $\triangle AA_1B$ is also right, the median A_1C_1 is half of the hypotenuse AB. In other words,

$$AB = 2A_1C_1 = 2s = AC,$$

completing the proof. ■

Problem 4

Let S be a finite set of **nonzero** real numbers, and let $f : S \to S$ be a function with the following property: for each $x \in S$, either

$$f(f(x)) = x + f(x) \qquad \text{or} \qquad f(f(x)) = \frac{x + f(x)}{2}.$$

Prove that $f(x) = x$ for all $x \in S$.

Solution

We will use the notation $f^n(x)$ to denote $f(f(\cdots f(x)\cdots))$, where we iterate the function n times. Suppose, to the contrary, that $f(x) \neq x$ for some $x \in S$. This implies that $f(f(x)) \neq f(x)$ as well, since $f(f(x))$ is either the sum or the average of x and $f(x)$ and these are distinct non-zero real numbers. Likewise, $f(f(x)) \neq f(x)$ implies that $f^3(x) \neq f(f(x))$. We can keep iterating the function to create a sequence

$$x, f(x), f(f(x)), f^3(x), f^4(x)\cdots,$$

where no term is equal to the term preceding it. However, since S is finite, eventually there has to be a repeating value. In other words, there exists m, n, with $n > m+1$, such that $f^m(x) = f^n(x)$.

Let $a = f^m(x)$. Then $a, f(a), f(f(a)), \cdots, f^{n-m}(a) = a$ is a *cycle* of length $n-m$. Since $f(f(x))$ cannot equal x (the sum of x and $f(x)$ cannot equal x since $f(x)$ is nonzero and the average of x and $f(x)$ cannot equal x because $f(x) \neq x$), the cycle has length at least 3. Since the cycle is finite, and the terms are nonzero, there must be a term of maximum absolute value. Call this M, and without loss of generality, assume that M is positive.

We know that the cycle has at least three terms, so consider the consecutive terms U, V, M in the cycle (since it is a cycle, it can start "anywhere"). We have $V = f(U)$ and $M = f(V) = f(f(U))$. We claim that V is positive, for if it were negative, then M would be either the average of U and V or the sum of U and V, which would force U to be larger than M, contradicting the fact that M is the largest term in the cycle.

But if V is positive, then $f(M) = f(f(V))$ must be greater than $M/2$, since it is either the sum or average of a positive number and M. Likewise, $f(f(M))$ must also be greater than $M/2$, since it is either the sum or average of M and a value that is greater than $M/2$. Once we have two consecutive terms in the cycle that are greater than $M/2$, all subsequent terms in the cycle will be greater than $M/2$. In other words, the cycle starting at M,

$$M, f(M), f(f(M)), f^3(M), \cdots$$

consists entirely of terms whose value is greater than $M/2$. Also, starting with the third term, each term is either the sum or average of the two terms preceding it. But since it is a cycle, eventually it will come back to the value of M, and that is impossible: M is neither the sum nor the average of two terms greater than $M/2$. We have achieved a contradiction, and conclude that there are no $x \in S$ such that $f(x) \neq x$; i.e. $f(x) = x$ for all $x \in S$.

Problem 5

Every positive integer is either *nice* or *naughty*, and the Oracle of Numbers knows which are which. However, the Oracle will not directly tell you whether a number is nice or naughty. The only questions the Oracle will answer are questions of the form "What is the sum of all nice divisors of n?," where n is a number of the questioner's choice. For instance, suppose (*just* for this example) that 2 and 3 are nice, while 1 and 6 are naughty. In that case, if you asked the Oracle, "What is the sum of all nice divisors of 6?," the Oracle's answer would be 5.

Show that for any given positive integer n less than 1 million, you can determine whether n is nice or naughty by asking the Oracle at most four questions.

Solution

Let $f(u)$ denote the sum of all nice divisors of u. Note that $f(u) \leq \sigma(u)$, where $\sigma(u)$ denotes the sum of all the divisors of u, including 1 and u.

We also note the following facts about the σ function:

1. σ is *multiplicative*: if a and b are relatively prime (share no prime factors), then $\sigma(ab) = \sigma(a)\sigma(b)$.
2. If p is prime, then $\sigma(p^a) = 1 + p + p^2 + \cdots + p^a = \frac{p^{a+1}-1}{p-1} = p^a\left(\frac{p-1/p^a}{p-1}\right) < p^a\left(\frac{p}{p-1}\right)$.

To solve the problem, let us consider a few cases.

- Suppose n is prime. Then ask the Oracle to compute $f(n)$. Clearly $f(n) \geq n$ if and only if n is nice.
- Suppose n is a power of a prime, so $n = p^a$ for some prime p, with $a > 1$. Again, ask the Oracle to compute $f(n)$. We claim again that $f(n) \geq n$ if and only if n is nice. Clearly if n is nice, then $f(n) > n$. But if n is naughty, then $f(n) = f(p^{a-1}) \leq \sigma(p^{a-1}) < p^{a-1}\left(\frac{p}{p-1}\right) \leq 2p^{a-1} \leq p^a$.
- Now suppose that n has at least two prime factors, so we can write $n = p^a q^b R$, where p, q are the two smallest primes found in the factorization of n, $a, b > 0$, and R is an integer whose prime factorization does not contain p or q (of course, R could equal 1). Note that R contains at most 5 different primes, since the product of the first 8 primes is larger than 1 million. We will modify the previous strategy, where we examined a subset of the divisors of n that were less than n and whose sum was bounded below n; this allowed us to focus on n and determine whether it is nice or naughty.

  The subset of the divisors that we seek is the set D of divisors of n that are multiples of $p^a q^b$; in other words, $D := \{p^a q^b r : r \text{ divides } R\}$. We claim that the sum of the elements of D is less than $2n$, and since $p^a q^b R = n \in D$, we conclude that n is nice if and only if the sum of the nice members of D is greater than or equal to n (this follows because our claim implies that the members of D that are not n have a sum that is strictly less than n).

Two things remain: to verify the claim that the sum of all the members of D is less than $2n$, and we need to know how to get the Oracle to compute the sum of the *nice* members of D with at most four questions.

1. The sum of the members of D is equal to $p^a q^b \sigma(R)$. By fact #2 above,

$$\sigma(R) < R\left(\frac{p_1}{p_1 - 1}\right)\left(\frac{p_2}{p_2 - 1}\right)\cdots,$$

   where the p_i range over the distinct prime factors of R. We know that there are at most five such prime factors, and this product is bounded above by

$$\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\left(\frac{11}{10}\right)\left(\frac{13}{12}\right)\left(\frac{17}{16}\right) = \frac{7 \cdot 11 \cdot 13 \cdot 17}{2^{10} \cdot 9} = \frac{1001 \cdot 17}{1024 \cdot 9},$$

   which is clearly less than 2.

2. Finally, let the "universal set" be the divisors of n, and let P and Q denote the divisors of n/p and n/q, respectively. It is easy to see that $P \cup Q$ is the complement of D (since membership in P guarantees that the exponent of p is too small, and membership in Q does the same for q). By the inclusion-exclusion principle, the members of D can be obtained by starting with the universal set (all divisors of n), removing members of P, removing members of Q, and then adding back in the members of $P \cap Q$, which just consists of the divisors of $n/(pq)$. Consequently, the sum of the *nice* members of D is given by

$$f(n) - f(n/p) - f(n/q) + f(n/(pq)),$$

   which requires just four questions to compute!