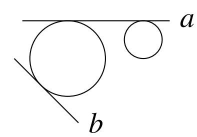
24th Bay Area Mathematical Olympiad

Problems and Solutions

Mar 1, 2023

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

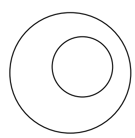
A A *tangent line* to a circle is a line that intersects the circle in exactly one point. A *common tangent line* to two circles is a line that is tangent to both circles. As an example, in the figure to the right, line a is a common tangent to both circles, but line b is only tangent to the larger circle.



Given two distinct circles in the plane, let n be the number of common tangent lines that can be drawn to these two circles. What are all possible values of n? Your answer should include drawings with explanations.

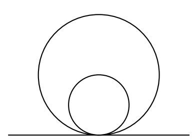
Solution: The possible values of n are 0, 1, 2, 3, and 4. These cases are illustrated below.

$n = 0$



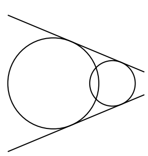
Two nonintersecting circles, one contained inside the other

$n = 1$



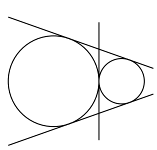
Two circles internally tangent to each other

$n = 2$



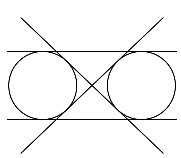
Two circles intersecting at two points

$n = 3$



Two circles externally tangent to each other

$n = 4$

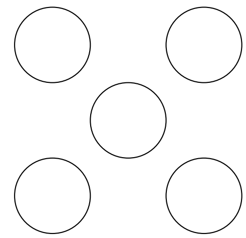


Two nonintersecting circles, neither contained inside the other

B Ara and Bea play a game where they take turns putting numbers from 1 to 5 into the cells of the X-shaped diagram on the right. Each number must be played exactly once, and a cell cannot have more than one number placed in it. Ara's goal is for the two diagonals of the X diagram to have the same sum when the game is over; Bea's goal is for these two sums to be unequal.

(a) Show that Ara can always win if he goes first.

(b) Show that Bea can always win if she goes first.



Solution:

(a) Suppose Ara goes first. He can begin by placing a 5 in the center of the board. This is the only space shared by both diagonals, so when the game is over, the two diagonal sums are guaranteed to add up to $1+2+3+4+5+5 = 20$.

For her first move, Bea must play some number n in one of the four corner spaces, where $n = 1$, 2, 3, or 4. Ara can then respond by playing $5-n$ in the opposite corner, completing a diagonal. (We know this move is still available, because $5-n$ is either 1, 2, 3, or 4, and cannot be equal to n.)

After these three moves, the sum of the completed diagonal is $5+n+(5-n) = 10$. Since the two diagonal sums will add up to 20 at the end of the game, the other diagonal sum will also be 10. Thus Ara will win.

(b) Suppose Bea goes first. She can begin by placing a 2 in the center of the board. By the same reasoning as in part (a), the two diagonal sums are now guaranteed to add up to $1+2+3+4+5+2 = 17$ at the end of the game.

For the diagonal sums to be equal, each diagonal sum would have to be $\frac{17}{2}$. But the diagonal sums are integers, so this cannot happen. Therefore, no matter what the remaining moves are, Bea will win.

Note that Ara's strategy also works with starting with 1 or 3 (because they are odd). Likewise, Bea can start with 2 or 4.

C/1 Mr. Murgatroyd decides to throw his class a pizza party, but he's going to make them hunt for it first. He chooses eleven locations in the school, which we'll call $1, 2, \ldots, 11$. His plan is to tell students to start at location 1, and at each location n from 1 to 10, they will find a message directing them to go to location $n+1$; at location 11, there's pizza!

Mr. Murgatroyd sends his teaching assistant to post the ten messages in locations 1 to 10. Unfortunately, the assistant jumbles up the message cards at random before posting them. If the students begin at location 1 as planned and follow the directions at each location, show that they will still get to the pizza.

Solution: If the students never visit the same room twice, then their hunt lasts a finite number of steps. In that case, they must reach the pizza (since the hunt always continues if they have not yet reached the pizza).

Therefore, the only way for the students to *not* reach the pizza is for them to visit the same room twice, which gets them stuck in a loop. Such a loop must consist of n rooms containing n messages that collectively point to that set of n rooms. But this means that all the messages pointing into the loop are in rooms that are part of the loop, so there's no way to enter the loop from outside.

Room 1 can't be part of a loop, since no message points to room 1. Thus, the students do not begin in a loop. Since they cannot enter a loop, they eventually get to the pizza. (In fact, they get to the pizza at least as quickly as Mr. Murgatroyd intended, since the worst case is that they have to visit every room once!)

D/2 Given a positive integer N (written in base 10), define its *integer substrings* to be integers that are equal to strings of one or more consecutive digits from N, including N itself. For example, the integer substrings of 3208 are 3, 2, 0, 8, 32, 20, 320, 208, and 3208. (The substring 08 is omitted from this list because it is the same integer as the substring 8, which is already listed.)

What is the greatest integer N such that no integer substring of N is a multiple of 9? (Note: 0 is a multiple of 9.)

Solution: The answer is 88,888,888.

In our solution, we'll make use of the well-known fact that an integer is divisible by 9 if and only if the sum of its digits (in base 10) is divisible by 9. It was permissible to use this fact without proof on the contest, but for the sake of completeness, a proof can be found in the appendix following this solution.

No integer substring of 88,888,888 is divisible by 9, since 9 does not divide $8k$ for any $k = 1, \ldots, 8$.

We now show that every $N > 88{,}888{,}888$ has an integer substring divisible by 9.

Suppose $N > 88{,}888{,}888$. If N has 8 digits, then one of those digits must be 9, which constitutes an integer substring by itself, so we are done. Thus, we assume from now on that N has 9 or more digits.

We claim that for any such N, there is some integer substring divisible by 9. In fact, we will describe an algorithm to find such a substring, using $N = 328{,}346{,}785$ as an illustrative example. For $0 \le k \le 9$, let s_k be the sum of the first k digits of N, where we define s_0 to be 0. We can think of s_k as a "running total" of the digits; here is our example number with $s_0, \ldots, s_9$ written below it:

$$\begin{array}{cccccccccc} & 3 & 2 & 8 & 3 & 4 & 6 & 7 & 8 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 3 & 5 & 13 & 16 & 20 & 26 & 33 & 41 & 46 \end{array}$$

Next we consider the remainders left by $s_0, \ldots, s_9$ when they are divided by 9. We have ten remainders with only nine possible values. By the pigeonhole principle, some two remainders must be equal. For instance, in our example, two of the remainders are equal to 5:

$$\begin{array}{cccccccccc} & 3 & 2 & 8 & 3 & 4 & 6 & 7 & 8 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 3 & 5 & 13 & 16 & 20 & 26 & 33 & 41 & 46 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 3 & \boxed{5} & 4 & 7 & 2 & 8 & 6 & \boxed{5} & 1 \end{array}$$

Suppose it is s_j and s_k that leave the same remainder (where $j < k$). Then $s_k - s_j$ is divisible by 9. But $s_k - s_j$ is the sum of the digits of the integer substring consisting of the $(j+1)^{\text{th}}$ through k^{th} digits of N. In the example above, for instance, this substring is 834678:

$$\begin{array}{cccccccccc} & 3 & 2 & \overline{8} & \overline{3} & \overline{4} & \overline{6} & \overline{7} & \overline{8} & 5 \\ & & \downarrow & & & & & & \downarrow & \\ & & 5 & & & & & & 41 & \end{array}$$

and we have $8+3+4+6+7+8 = 41-5 = 36$, a multiple of 9.

Since we have an integer substring whose digits add up to a multiple of 9, that substring is itself a multiple of 9, and we are finished.

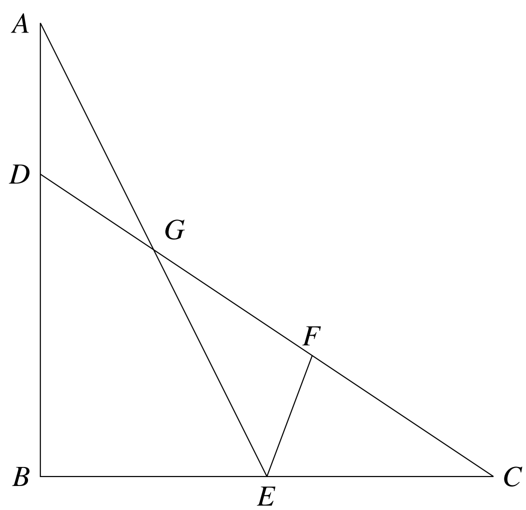
Appendix. Here is the proof of the "well-known fact" mentioned at the beginning. To keep the notation simple, we state the proof for 4-digit integers; it should be clear how to generalize to integers with any number of digits.

Let $n = \underline{abcd}$, where the underline means we are writing the digits of n. By the nature of base 10 representation, we have

$$\begin{aligned} n &= 1000a + 100b + 10c + d \\ &= (999+1)a + (99+1)b + (9+1)c + d \\ &= (999a + 99b + 9c) + (a+b+c+d). \end{aligned}$$

The first bracketed quantity is a multiple of 9. Thus, n and $a+b+c+d$ differ by a multiple of 9. In particular, if n is a multiple of 9, then so is $a+b+c+d$ (the sum of its digits), and vice versa. This completes our proof.

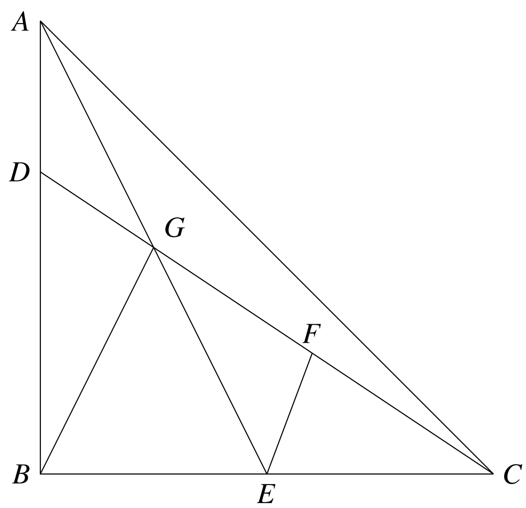
E/3 In the following figure—not drawn to scale!—E is the midpoint of BC, triangle FEC has area 7, and quadrilateral $DBEG$ has area 27. Triangles ADG and GEF have the same area, x. Find x.



Solution: The answer is $x = 8$.

Use the notation $[\cdot]$ to denote the area of a polygon. Draw GB; notice that triangles GBE and GEC have equal bases and altitudes, so $[GBE] = [GEC] = x + 7$. Since $[ABE] = 27 + x$, we have $[GDB] = 20 - x$.

Likewise, if we draw AC, we see that $[ABE] = [AEC] = 27 + x$, so $[AGC] = 20$, which implies that $[CAD] = 20 + x$.



Now triangles GAD and GDB have the same altitude (from G to AB), so their bases are proportional to their respective areas. In other words,

$$\frac{AD}{DB} = \frac{[GAD]}{[GDB]} = \frac{x}{20-x}.$$

But AD and DB are also the bases of triangles CAD and CBD, which have the same altitude (from C to AB). Hence

$$\frac{AD}{DB} = \frac{[CAD]}{[CDB]} = \frac{20+x}{34+x}.$$

Equating these two fractions leads to the quadratic equation $34x + x^2 = 400 - x^2$; the only positive solution is $x = 8$.

4 Zaineb makes a large necklace from beads labeled $290, 291, 292, \ldots, 2023$. She uses each bead exactly once, arranging the beads in the necklace any order she likes. Prove that no matter how the beads are arranged, there must be three beads in a row whose labels are the side lengths of a triangle.

Solution: More generally, we will prove that if there are $6n$ beads labeled $n+1, n+2, \ldots, 7n$, there must be three beads in a row whose labels are the side lengths of a triangle. (When $n = 289$, this coincides with the problem statement.)

Aiming for a contradiction, assume there are no three beads in a row whose labels are the side lengths of a triangle.

By starting at an arbitrary position on the necklace and counting off three beads at a time, partition the $6n$ beads into $2n$ trios of consecutive beads. Let S be the sum obtained by adding together the smallest two numbers from every trio. (Thus, S is a sum of $4n$ numbers.) Let T be the sum obtained by adding the largest number from every trio.

By our assumption, within each trio, the sum of the two smallest numbers is less than or equal to the largest number. By adding these inequalities across all trios, we see that $S \leq T$.

On the other hand, S can be no smaller than the sum of the $4n$ smallest numbers. Using the formula for the sum of an arithmetic progression, we have

$$S \geq (n+1) + (n+2) + \cdots + (5n) = \frac{(4n)(6n+1)}{2} = 12n^2 + 2n.$$

Similarly, T can be no larger than the sum of the $2n$ largest numbers:

$$T \leq (5n+1)+(5n+2)+\cdots+(7n) = \frac{(2n)(12n+1)}{2} = 12n^2+n.$$

Thus $T \leq 12n^2+n < 12n^2+2n \leq S$, which contradicts our earlier claim that $S \leq T$.

We have arrived at a contradiction, so there must in fact be three beads in a row whose labels are the side lengths of a triangle.

5 A *lattice point* in the plane is a point with integer coordinates. Let T be a triangle in the plane whose vertices are lattice points, but with no other lattice points on its sides. Furthermore, suppose T contains exactly four lattice points in its interior. Prove that these four points lie on a straight line.

Solution: Let us begin with some preliminaries. In the solution to follow, we treat points freely as vectors, e.g. writing nA to mean the point whose coordinates are n times the coordinates of A, or $A+B$ to mean the point which is the coordinate-wise sum of A and B.

A basic result from vector geometry, which we will assume, states that given any three noncollinear points A,B,C in the plane, every point Q may be represented in the form $rA+sB+tC$ for unique r,s,t satisfying $r+s+t=1$. Moreover, Q is in the interior of $\triangle ABC$ if and only if such r,s,t are all positive. In similar fashion, any point on the line through A and B can be expressed as $rA+sB$ with $r+s=1$, and lies between A and B if and only if $r,s>0$.

We will also make repeated use of Pick's Theorem, which we now state without proof. This theorem asserts that a lattice polygon (a polygon whose vertices are lattice points) has area equal to

$$i+\frac{1}{2}b-1,$$

where i and b are the number of lattice points on the polygon's interior and boundary, respectively. Thus (for instance), the triangle T described in the problem must have area $4+\frac{1}{2}(3)-1=\frac{9}{2}$.

Now we are ready to begin the solution. With no loss of generality, let us assume T has one vertex at the origin O, which we identify with the zero vector. Call the other two vertices A and B.

Of the four lattice points in the interior of T, let P be the point closest to line OA. It follows that there are no lattice points lying inside $\triangle OPA$ or on its boundary, other than O,P,A themselves, since any such point would be closer than P to line OA. Therefore, by Pick's Theorem, $\triangle OPA$ has area $\frac{1}{2}$.

Lemma: Every lattice point Q can be expressed in the form $nP+kA$ for some pair of integers (n,k). Moreover, when Q is expressed in such form, we have $n=2[OQA]$. (The brackets represent area.)

Proof. Let Q be a lattice point. By Pick's Theorem, $[OQA]=\frac{n}{2}$ for some integer n. Thus $[OQA]=n\cdot[OPA]$. By the base–height formula for triangle area, it follows that Q is on the line parallel to line OA that passes through the point nP. Thus $Q=nP+kA$ for some real k, where kA is a lattice point.

We assert that k is an integer. Indeed, if $\{k\}$ denotes the fractional part of k, then $\{k\}A=kA-\lfloor k\rfloor A$ is a lattice point which lies on segment OA, part of the boundary of T. Since T has no lattice points on its boundary other than its vertices, we must have $\{k\}=0$. This completes the proof of the lemma.

Let us return to the main problem. As already noted, $[T]=[OBA]=\frac{9}{2}$. Thus by the lemma, $B=9P-kA$ for some integer k (the minus sign in the expression is not a typo, but a deliberate convenience for what follows). Rearranging, and using the fact that O is the zero vector, we have $P=\frac{k}{9}A+\frac{1}{9}B+\frac{8-k}{9}O$. Since P is in the interior of T, we have $0<k<8$. We will consider the possible values of k in turn.

If $k\equiv 0 \pmod 3$, then $\frac{1}{3}B=3P-\frac{k}{3}A$ is a lattice point lying on segment OB. This contradicts the specification of T as having no lattice points on its sides.

If $k \equiv 2 \pmod 3$, then $\frac{1}{3}B + \frac{2}{3}A = 3P - \frac{k-2}{3}A$ is a lattice point lying on AB, similarly yielding a contradiction.

The remaining possibilities are $k = 1, 4, 7$.

If $k = 1$, then the interior of T contains in its interior the four collinear lattice points

$$P = \tfrac{1}{9}A + \tfrac{1}{9}B + \tfrac{7}{9}O, \quad 2P = \tfrac{2}{9}A + \tfrac{2}{9}B + \tfrac{5}{9}O, \quad 3P = \tfrac{3}{9}A + \tfrac{3}{9}B + \tfrac{3}{9}O, \quad 4P = \tfrac{4}{9}A + \tfrac{4}{9}B + \tfrac{1}{9}O.$$

If $k = 4$, then the interior of T contains in its interior the four collinear lattice points

$$P = \tfrac{4}{9}A + \tfrac{1}{9}B + \tfrac{4}{9}O, \quad 3P - A = \tfrac{3}{9}A + \tfrac{3}{9}B + \tfrac{3}{9}O, \quad 5P - 2A = \tfrac{2}{9}A + \tfrac{5}{9}B + \tfrac{2}{9}O, \quad 7P - 3A = \tfrac{1}{9}A + \tfrac{7}{9}B + \tfrac{1}{9}O.$$

If $k = 7$, then the interior of T contains in its interior the four collinear lattice points

$$P = \tfrac{7}{9}A + \tfrac{1}{9}B + \tfrac{1}{9}O, \quad 2P - A = \tfrac{5}{9}A + \tfrac{2}{9}B + \tfrac{2}{9}O, \quad 3P - 2A = \tfrac{3}{9}A + \tfrac{3}{9}B + \tfrac{3}{9}O, \quad 4P - 3A = \tfrac{1}{9}A + \tfrac{4}{9}B + \tfrac{4}{9}O.$$

Thus, the four lattice points inside T are collinear in every case, as desired.

Solution 2: We assume the same basic facts about vectors as in the previous solution, as well as Pick's Theorem and the determinant formula for the area of a parallelogram.

Let T have vertices $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$. We know that

$$\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = 2[ABC] = 9. \tag{1}$$

Consider equation (1) modulo 3, that is, over the field $\mathbb{Z}/3\mathbb{Z}$. In this setting, the determinant is zero, so the vectors $\mathbf{u} = (x_2 - x_1, y_2 - y_1)$ and $\mathbf{v} = (x_3 - x_1, y_3 - y_1)$ are linearly dependent. If either of these vectors is zero (mod 3, that is), or if they are equal, then the trisection points of a side of T are lattice points, which contradicts the problem statement. Thus $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ and $\mathbf{u} = -\mathbf{v}$.

An immediate consequence is that $(x_1 + x_2 + x_3, y_1 + y_2 + y_3) = \mathbf{u} + \mathbf{v} + 3(x_3, y_3) = \mathbf{0}$ over $\mathbb{Z}/3\mathbb{Z}$, with the result that the centroid, $G = \frac{1}{3}(A + B + C)$, is a lattice point.

Now consider $\triangle ABG$, whose area is $\frac{1}{3}[ABC] = \frac{3}{2}$. By Pick's Theorem, $\triangle ABG$ has either

- one lattice point in its interior and none on its boundary (besides vertices), or
- two lattice points on its boundary.

Case 1: $\triangle ABG$ has a lattice point in its interior and none on its boundary. In this case, a repetition of the preceding (mod 3) argument shows that the centroid G_1 of $\triangle ABG$ is a lattice point. In this case, $G_1 + k(G - G_1)$ for $k = 0, 1, 2, 3$ are four collinear lattice points inside T.

Case 2: $\triangle ABG$ has two lattice points on its boundary. Note that if at least two lattice points occur on a line, then the lattice points on that line occur at regular intervals. Thus the two lattice points on the boundary of $\triangle ABG$ are either the midpoints of AG and BG or the trisection points of one of these sides (say, AG). In the two cases, if we extend side AG beyond G, the next lattice point occurring on the extension is respectively either **on** T (at the midpoint of side BC), which is a contradiction, or **inside** T, being then the fourth collinear lattice point inside T. So we are finished.

Alternative solutions (sketches): One participant used the simplifying idea of an *affine transformation*. Start by assuming the that one vertex of the triangle is at the origin $(0,0)$ and another at the point (m,n), where m and n are relatively prime (why?). Use this and Pick's theorem to show that we can find integers p, q so that the linear transformation that maps (x,y) to $(nx - my, px + qy)$ will transform our triangle to one with vertices at $(0,0)$ and $(0,1)$, with the third vertex having x-coordinate of 9. We now have a (fairly) simple triangle with a number of relatively simple cases to examine.

Another participant used *barycentric coordinates*, where the coordinates of points in or on the triangle is a weighted average of the coordinates of the vertices. This gives convenient ways to characterize colinear points, among other things.