27th Bay Area Mathematical Olympiad

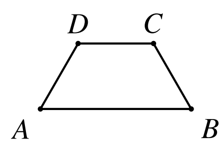
Problems and Solutions

February 25, 2026

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

Problems

A Let $\mathscr{T} = ABCD$ be an isosceles trapezoid with side lengths $1, 1, 1$, and 2. ("Isosceles" means that angles A and B in the figure below are equal.)



(a) Draw a diagram that illustrates how to dissect $\mathscr{T}$ into four congruent trapezoids, each similar to $\mathscr{T}$. In other words, we divide $\mathscr{T}$ into four congruent trapezoids that do not overlap and have no gaps between them, and each is a scaled-down version of $\mathscr{T}$.

(b) Do the same as (a), but now dissect $\mathscr{T}$ into nine congruent trapezoids, each similar to $\mathscr{T}$.

Draw your diagrams carefully, so that it is clear how your dissections work.

B Suppose that

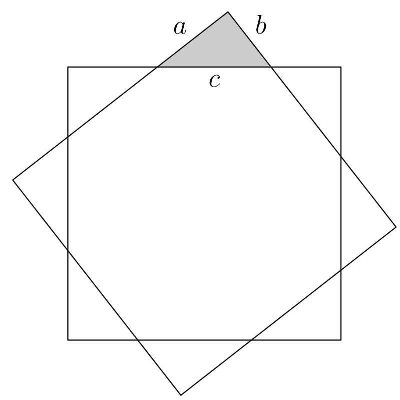
- a is a positive real number,
- b is the greatest integer less than or equal to a,
- c is the greatest integer less than or equal to ab, and
- $ac = 42$.

Is there any value of a for which this occurs? If so, what are all such value(s), and how do you know there are no others?

(If you are familiar with "floor" notation, this problem is asking you to solve the equation $a\lfloor a\lfloor a\rfloor\rfloor = 42$.)

C/1 Let B be a brick (that is, a rectangular prism). This particular brick has the property that a single cut can split B into two congruent bricks that are similar to B. In other words, these smaller bricks are scaled-down copies of the original brick. Find possible dimensions for B (length, width, height) and explain your answer.

D/2 Two distinct squares overlap to form an octagon and eight right triangles (as shown in the diagram). The squares each have side length 4 and have the same center. If one of those right triangles has side lengths a, b, c, where c is the hypotenuse, show that the area of that triangle is $a + b - c$.



E/3 Lou is in a logic class with six other students. Their teacher assigned each of them the role of *Truth-Teller* or *Liar*. The Truth-Tellers must make only truthful statements, and the Liars must make only false statements. Each student—except Lou, who was daydreaming when the roles were assigned—knows their own role and that of all the other students. The teacher asks the students to sit in a circle, where they will be asked a question, and they must answer "in character." Lou doesn't know her own role nor anybody else's.

The teacher asks each student in the circle, one after another, "Is the person sitting to your left a Truth-Teller?" to which they must answer "Yes" or "No." Lou happens to be the last person to be called. She heard "Yes" three times and "No" three times from the other students, in some order. Suddenly Lou realizes that she can act out her role correctly, even though she doesn't know what her role is! When the teacher asks Lou the same question as the others, what should her answer be, and why?

4 A large blackboard initially has the single number 2026 written on it. Fran writes new numbers on the blackboard by following these rules:

- If the number $a^2 + b^2$ is written on the board, where a, b are distinct positive integers, then Fran can write both a^2 and b^2 on the board.
- If the two numbers a^2 and b^2 are both written on the board, where a, b are distinct positive integers, then Fran can write $a^2 + b^2$ on the board.

Proceeding in this way, will Fran ever be able to write the number 2026^2 on the board? What about the number 2026^{2026}?

5 Given a deck of cards numbered 1 through N ($N > 1$), not necessarily arranged in order, we define a *grand swap* as the following N-step procedure: swap the part of the deck that's above the 1 with the part that's below the 1; then swap the part of the deck that's above the 2 with the part that's below the 2; continue in this manner, until finally swapping the part of the deck that's above the N with the part that's below the N. (If one of the "parts" that we swap is empty, the swap is equivalent to simply moving the top card to the bottom or vice versa.)

For example, suppose we start with $1, 2, 4, 3$ (with 1 on the top and 3 on the bottom). We swap the cards above the 1 (all zero of them) with the cards below the 1 (all three of them) to get $2, 4, 3, 1$. Then we swap the cards above the 2 with the cards below the 2 to get $4, 3, 1, 2$. Then we swap the card above the 3 with the cards below the 3 to get $1, 2, 3, 4$ (notice that the 1 and the 2 remain in the order they were in above; though two piles are swapped, each pile stays in its original order). Finally we swap the cards above the 4 with the cards below the 4 to get $4, 1, 2, 3$. So, our grand swap sends $1, 2, 4, 3$ to $4, 1, 2, 3$.

Show that performing $N+1$ successive grand swaps always brings the deck back to its original order.

Problems and Solutions

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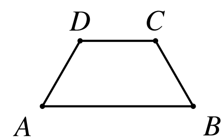
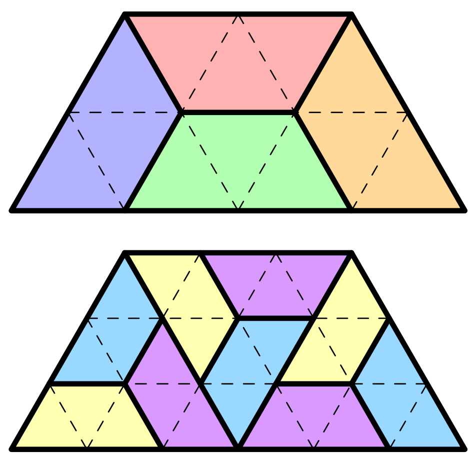


(a) Draw a diagram that illustrates how to dissect $\mathscr{T}$ into four congruent trapezoids, each similar to $\mathscr{T}$. In other words, we divide $\mathscr{T}$ into four congruent trapezoids that do not overlap and have no gaps between them, and each is a scaled-down version of $\mathscr{T}$.

(b) Do the same as (a), but now dissect $\mathscr{T}$ into nine congruent trapezoids, each similar to $\mathscr{T}$.

Draw your diagrams carefully, so that it is clear how your dissections work.

Solution: Here are diagrams illustrating possible dissections into four and nine trapezoids. The dissections have been overlaid on a grid of equilateral triangles, helping to illustrate why they work. Note that the dissection into nine trapezoids is not unique.



Following is a more formal description of these dissections (although this level of formality was not expected of contestants).

(a) Let $ABCD$ be the original trapezoid, with AB and CD as the longer and shorter bases, respectively.

Draw a line ℓ parallel to and equidistant from the two bases.

On ℓ, place two points E and F that are symmetric with respect to the vertical line through the midpoints of AB and CD, and $\frac{1}{2}$ unit apart.

This construction yields four sub-trapezoids: $EDCF$; its reflection across ℓ; and the two remaining disjoint regions of $ABCD$, which are themselves trapezoids.

We may readily check that the four sub-trapezoids are each similar to $ABCD$ at $\frac{1}{2}$ scale.

(b) Again, let $ABCD$ be the original trapezoid, with AB and CD as the longer and shorter bases. Let E be the midpoint of AB. Connect E to D and C. This divides $ABCD$ into three congruent equilateral triangles: ADE, EDC, and CBE.

Each triangle can be subdivided into three congruent trapezoids similar to $ABCD$. We illustrate this for ADE. Each side of ADE has length 1.

Define three points:

- A_1 on AD, one-third of the way from A to D;
- D_1 on DE, one-third of the way from D to E;
- E_1 on EA, one-third of the way from E to A.

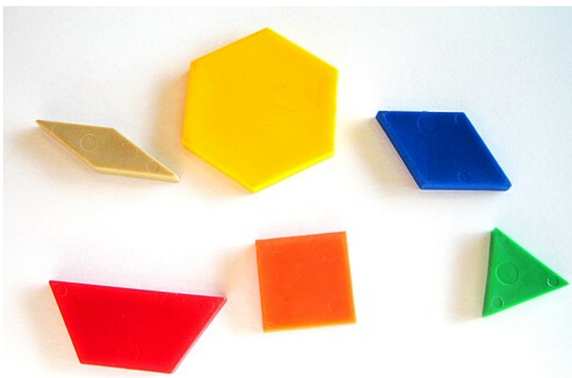
Draw three lines:

- through A_1 parallel to AE;
- through D_1 parallel to AD;
- through E_1 parallel to DE.

These three lines cross at one point O, which is the center of the equilateral triangle ADE. The three sub-trapezoids are AA_1OE_1, A_1DD_1O, and E_1OD_1E.

We may readily check that the nine sub-trapezoids are each similar to $ABCD$ at $\frac{1}{3}$ scale.

Remark. Instead of dissecting a trapezoid into smaller trapezoids, one can also conceive of the problem in terms of using trapezoidal blocks to assemble a *larger* trapezoid. This is equivalent to the original problem, but may be easier to think about... especially if one remembers playing with these blocks in elementary school:[1]



Shapes which can be dissected into similar copies of themselves, like the trapezoid in this problem, are called *rep-tiles*. The reader is invited to show that the trapezoid can be dissected into n similar trapezoids (not necessarily all the same size) for every $n \geq 15$.

B Suppose that

- a is a positive real number,
- b is the greatest integer less than or equal to a,
- c is the greatest integer less than or equal to ab, and
- $ac = 42$.

Is there any value of a for which this occurs? If so, what are all such value(s), and how do you know there are no others?

[1]Image from Wikimedia Commons, used under Creative Commons license (link).

(If you are familiar with "floor" notation, this problem is asking you to solve the equation $a\lfloor a\lfloor a\rfloor\rfloor =$ 42.)

Solution: We can assume $a \geq 1$ (since if $a < 1$, then $b = 0$ and $c = 0$).

Suppose a has some existing value. If we increase that value, then b and c also increase or are unchanged, so ac must increase. This shows that there is at most one value of a for which ac hits the target value of 42. It also tells us that if we try a value of a and find that $ac < 42$, then we should increase a (or if $ac > 42$, we should decrease a). This will enable us to "home in" on the solution, if there is one.

If a is an integer, then $b = a$, $c = a^2$, and $ac = a^3$. Thus $a = 3$ is too small and $a = 4$ is too large, since

$$3^3 < 42 < 4^3.$$

Given that $3 < a < 4$, we have $b = 3$, and the possible values of c are 9, 10, and 11. Of these, only 11 is large enough to allow $ac = 42$ (given that $a < 4$). So we must have $c = 11$ and $a = \frac{42}{11}$.

Indeed, $a = \frac{42}{11}$ is a valid solution, since we then have

$$\begin{aligned} b &= 3, \\ ab &= \tfrac{126}{11} = 11\tfrac{5}{11}, \\ c &= 11, \end{aligned}$$

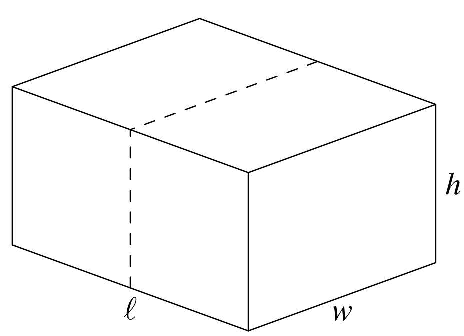
and $ac = 42$ as required. So the sole solution is $a = \frac{42}{11}$.

C/1 Let B be a brick (that is, a rectangular prism). This particular brick has the property that a single cut can split B into two congruent bricks that are similar to B. In other words, these smaller bricks are scaled-down copies of the original brick. Find possible dimensions for B (length, width, height) and explain your answer.

Solution: Let the dimensions of B be $\ell \times w \times h$, ordered so that $\ell \geq w \geq h$.

To cut B into two congruent smaller bricks (rectangular prisms), we can bisect B along a plane midway between any pair of parallel faces. There are three ways to make such a cut; we could bisect the length, width, or height of B.

However, for the cut to possibly produce two bricks similar to B, we must bisect the longest dimension of B. This is the only way for the longest side of the resulting smaller bricks to be shorter than the longest side of B (as it must be, if they are similar). Here is an illustration of this cut:



The two resulting bricks have dimensions $w \times h \times \frac{\ell}{2}$.

The *shortest* side of these bricks must also be shorter than the *shortest* side of B. Thus, $w \geq h \geq \frac{\ell}{2}$, so we now know which sides of the original brick and the smaller bricks must correspond under scaling. If the original brick and the smaller bricks are similar in the ratio $r:1$, then we have

$$\ell = rw,$$
$$w = rh,$$
$$h = r\ell/2.$$

Multiplying all three equations together gives

$$\ell wh = r^3 \ell wh/2.$$

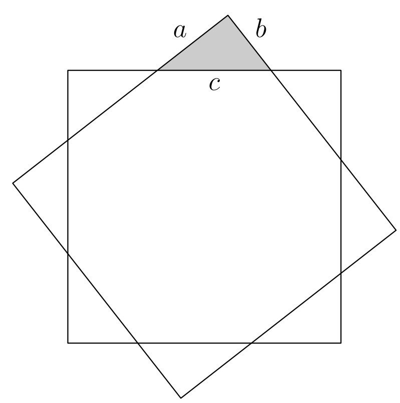
By canceling common factors from both sides, we obtain $r^3 = 2$, and so $r = \sqrt[3]{2}$. (We could also figure this out from the fact that the original brick has twice the volume of each smaller brick.)

Hence for any positive value of h, we may take $w = (\sqrt[3]{2})h$ and $\ell = (\sqrt[3]{2})w = (\sqrt[3]{4})h$, and an $\ell \times w \times h$ brick will have the desired property. For example, we may set $h = 1$ to obtain the possible dimensions

$$\sqrt[3]{4} \times \sqrt[3]{2} \times 1.$$

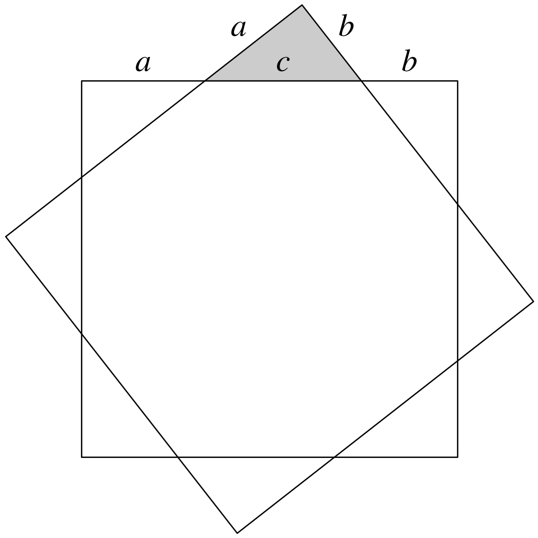
Remark. The problem was inspired by the international "A" series standard for paper sizes, which you can read about online.

D/2 Two distinct squares overlap to form an octagon and eight right triangles (as shown in the diagram). The squares each have side length 4 and have the same center. If one of those right triangles has side lengths a, b, c, where c is the hypotenuse, show that the area of that triangle is $a + b - c$.



Solution: Observe that all the right triangles are congruent. (One way to see this is that the whole diagram is unchanged under rotation by 90° around the squares' common center, or under a reflection that exchanges the two squares. Thus each of the eight triangles can be rigidly mapped to any other.)

Each side of each square is cut by the other square into lengths a, c, b (as seen below), so $a + b + c = 4$.



Now we may finish in a couple of ways.

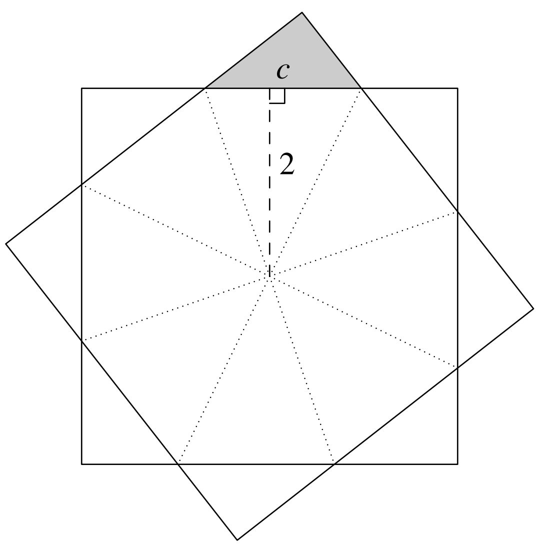
First approach. Since $a+b+c=4$, we have

$$\begin{aligned}4(a+b-c) &= (a+b+c)(a+b-c)\\ &= (a+b)^2-c^2\\ &= a^2+b^2+2ab-c^2\\ &= 2ab,\end{aligned}$$

where we have used the Pythagorean Theorem in the last step to write $a^2+b^2-c^2=0$.

The area of the shaded triangle is $\frac{1}{2}ab$. We see by the above work that this is equal to $a+b-c$.

Second approach. The octagon has perimeter $8c$. The octagon also has area $8c$, because connecting the center to the vertices decomposes it into eight triangles of height 2 and base c, as shown below.



Thus the area that is inside one given square, but outside the octagon, is $16-8c$, and each right triangle has area $\frac{1}{4}(16-8c)=4-2c$. We may substitute $4=a+c+b$ to write the area of the triangle as $a+b-c$.

Remark. The reader is invited to show that $b=\dfrac{8-4a}{4-a}$ and to write down a similar formula for c.

Thus when a is rational, so are b and c. These formulas can be used to generate the Pythagorean triples; for example, substitute $a = 1$ (and apply a scaling factor) to obtain a famous right triangle.

E/3 Lou is in a logic class with six other students. Their teacher assigned each of them the role of *Truth-Teller* or *Liar*. The Truth-Tellers must make only truthful statements, and the Liars must make only false statements. Each student—except Lou, who was daydreaming when the roles were assigned—knows their own role and that of all the other students. The teacher asks the students to sit in a circle, where they will be asked a question, and they must answer "in character." Lou doesn't know her own role nor anybody else's.

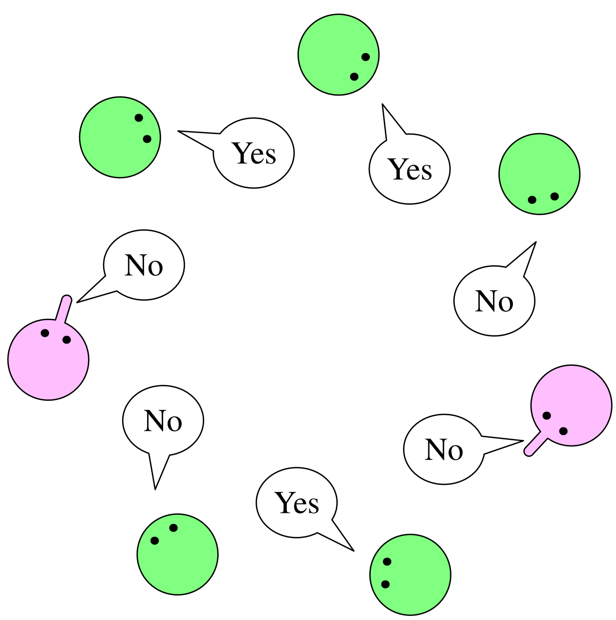
The teacher asks each student in the circle, one after another, "Is the person sitting to your left a Truth-Teller?" to which they must answer "Yes" or "No." Lou happens to be the last person to be called. She heard "Yes" three times and "No" three times from the other students, in some order. Suddenly Lou realizes that she can act out her role correctly, even though she doesn't know what her role is! When the teacher asks Lou the same question as the others, what should her answer be, and why?

Solution: When the teacher asks Person A the question, if Person B is to Person A's left, here is what happens:

Person A's role	Person B's role	Person A's response
Truth-Teller	Truth-Teller	Yes
Truth-Teller	Liar	No
Liar	Truth-Teller	No
Liar	Liar	Yes

Thus, when any person answers "Yes," what it really tells us is that they are on the same "team" as the person to their left (either both are Truth-Tellers or both are Liars). Meanwhile, a "No" answer tells us that they are on opposite teams.

For example, the diagram below shows one of many possible arrangements; Truth-Tellers are shown in green, Liars in pink (with protruding noses), and each person is "looking" at the person to their left.



As we look once around the circle, runs of one or more consecutive Truth-Tellers alternate with runs of one or more consecutive Liars. (In the example above, there are two runs of Truth-Tellers and two runs of Liars.) Therefore, for each Truth-Teller with a Liar to their left, there must be a Liar with a Truth-Teller to their left, and vice versa.

Thus, once all players have spoken, there should be an even number of “No” answers. Since Lou has heard “No” only three times, her proper response must be to provide the fourth “No.”

4 A large blackboard initially has the single number 2026 written on it. Fran writes new numbers on the blackboard by following these rules:

- If the number a^2+b^2 is written on the board, where a,b are distinct positive integers, then Fran can write both a^2 and b^2 on the board.
- If the two numbers a^2 and b^2 are both written on the board, where a,b are distinct positive integers, then Fran can write a^2+b^2 on the board.

Proceeding in this way, will Fran ever be able to write the number 2026^2 on the board? What about the number 2026^{2026}?

Solution: For those unfamiliar, the notation $a \equiv b \pmod{n}$, used extensively in this solution, means that $a-b$ is a multiple of n. (This notation is read as "a is congruent to b mod n.")

We will prove the following:

Claim 1. If $a \equiv 2 \pmod{4}$, then Fran can never write a^2 on the board.

Claim 2. If $a \not\equiv 2 \pmod{4}$, then Fran can write a^2 on the board.

Before proving these claims, we note that they provide the answers to the problem. Since $2026 \equiv 2 \pmod{4}$, Fran cannot write 2026^2 on the board. However, since $2026^{1013} \equiv 0 \pmod{4}$, Fran *can* write $(2026^{1013})^2 = 2026^{2026}$ on the board.

Proof of Claim 1.

First, note these facts:

- For every integer a, we have $a^2 \equiv 0$, 1, 4, or 9 (mod 16).
- If $a \equiv 2 \pmod{4}$, then $a^2 \equiv 4 \pmod{16}$, and vice versa.

These facts are easily checked by writing out the expansions of $(8k)^2$, $(8k\pm1)^2$, $(8k\pm2)^2$, $(8k\pm3)^2$, and $(8k+4)^2$; we omit the details.

We claim that these statements hold at all times:

I. Every number on the blackboard is congruent to 0, 1, 2, 9, or 10 (mod 16).

II. Every **square** number on the blackboard is congruent to 0, 1, or 9 (mod 16).

Statements I and II certainly hold at the beginning of the process, when the only number on the board is 2026 (which is congruent to 10 mod 16, and is not a square).

To see why statements I and II continue to hold at all times, consider this addition (mod 16) table:

	0	**1**	**9**	**4**
0	0	1	9	4
1	1	2	10	5
9	9	10	2	13
4	4	5	13	8

There are two ways Fran can write new numbers on the board. The first way is that if a^2+b^2 is on the board, Fran can write a^2 and b^2. This rule produces square numbers, which must be 0, 1, 4, or $9 \pmod{16}$. For this rule to produce a number congruent to $4 \pmod{16}$, the table shows that there would have to already be a number congruent to 4, 5, 13, or 8 on the board. This cannot happen as long as statements I and II hold.

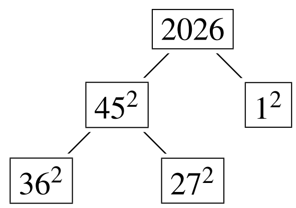
The second way for Fran to write new numbers is to write $a^2 + b^2$ if a^2 and b^2 are already present. But if all squares on the board are 0, 1, or 9 $\pmod{16}$, then the table shows that this rule can only produce new numbers of the form 0, 1, 2, 9, or 10 $\pmod{16}$.

Thus, we see that if statements I and II hold at some time, then they will continue to hold after Fran writes the next number. This shows that statements I and II hold at all times, which proves Claim 1.

Proof of Claim 2.

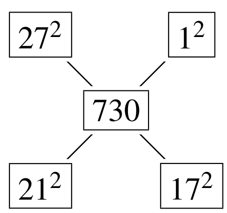
Now we will prove that for every $a \not\equiv 2 \pmod{4}$, Fran can eventually write a^2 on the board. We begin by getting some numbers on the board however we can; once we have enough, we'll show a systematic way to produce the rest *ad infinitum*.

Here are some opening moves:

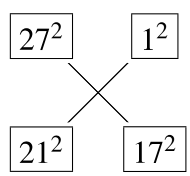


In other words: Since $2026 = 45^2 + 1^2$, write 45^2 and 1^2 on the board; then, since $45^2 = 36^2 + 27^2$, write 36^2 and 27^2 on the board.

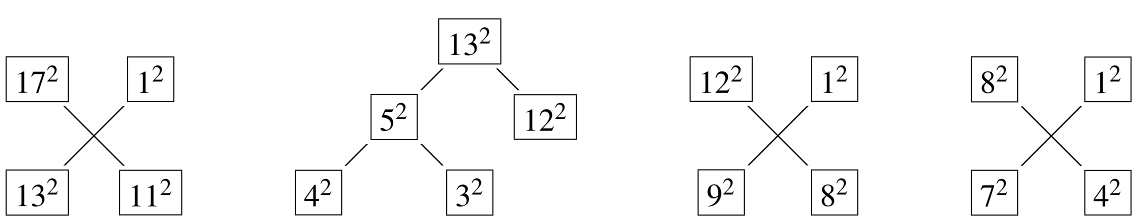
Next, Fran can do this:



In other words: Since 27^2 and 1^2 are already on the board, write their sum, 730, on the board. Then, since 730 is also equal to $21^2 + 17^2$, write 21^2 and 17^2. Since 730 is used merely as a "stepping stone", we may abbreviate the diagram above to

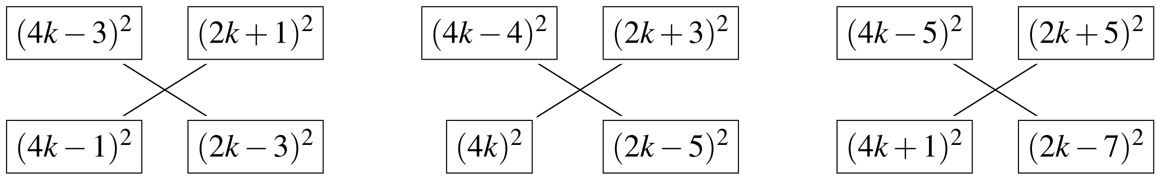


In general, if a^2 and b^2 are on the board and $a^2 + b^2 = c^2 + d^2$, Fran can write c^2 and d^2 on the board. Here are several more moves she can perform:



One may wonder *why* it is so frequently the case that a sum of two squares can be rewritten as a different sum of two squares. This is addressed in a remark following the solution.

Fran now has a^2 on the board for all $1 \le a \le 13$ with $a \not\equiv 2 \pmod 4$. We are now ready to generate the rest of the squares systematically, using the following moves:

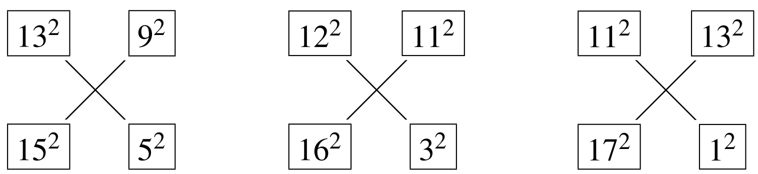


Each of these moves is justified by an identity, e.g. (for the first one)

$$(4k-3)^2 + (2k+1)^2 = (4k-1)^2 + (2k-3)^2.$$

We leave it to the reader to check these identities. (Suggestion: First rearrange so that each side is a *difference* of squares.)

To generate a^2 for all $a \not\equiv 2 \pmod 4$, Fran cycles through the three types of moves above for $k = 4$, then $k = 5$, $k = 6$, and so on. For example, on the first pass ($k = 4$), her moves look like this:



We see that Fran has produced 15^2, 16^2, and 17^2. The next pass ($k = 5$) produces 19^2, 20^2, and 21^2, and so on. At each step, the squares Fran needs to perform the next move are already on the board. This completes the proof of Claim 2, and with it the solution.

Remark. As was mentioned above, it is no coincidence that a sum of two squares can often be represented as a sum of two squares in a second way. A trick with complex numbers shows why, and also shows how to discover such representations without exhaustive computation. We illustrate for the number 730 (which appeared in our solution).

We will use a few facts about absolute value of complex numbers. First, $|a+bi|^2 = a^2 + b^2$. Second, $|zw| = |z||w|$ for complex numbers z, w. Finally, $|\overline{z}| = |z|$, where $\overline{a+bi}$ is the *complex conjugate* $a - bi$.

Observe that $730 = (73)(10) = (8^2+3^2)(3^2+1^2)$. Thus, $730 = |8+3i|^2|3+i|^2 = |(8+3i)(3+i)|^2$. We have $(8+3i)(3+i) = 21+17i$, so we can read off $730 = 21^2 + 17^2$.

On the other hand, we can also write $730 = |8+3i|^2|3-i|^2 = |(8+3i)(3-i)|^2 = |27+i|^2$, hence $730 = 27^2 + 1^2$. By replacing one of the complex factors with its conjugate, we obtain a different representation as a sum of two squares!

Generalizing these ideas, we may obtain the *Brahmagupta–Fibonacci identity*, which states

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= (ac+bd)^2 + (ad-bc)^2 \\ &= (ac-bd)^2 + (ad+bc)^2.\end{aligned}$$

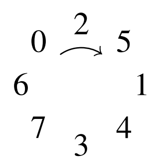
This identity was observed by Diophantus as early as the 3rd century C. E.; while the identity is not hard to *verify*, the arithmetic of complex numbers provides a deeper explanation which was not available to Diophantus (or Brahmagupta or Fibonacci).

5 Given a deck of cards numbered 1 through N ($N > 1$), not necessarily arranged in order, we define a *grand swap* as the following N-step procedure: swap the part of the deck that's above the 1 with the part that's below the 1; then swap the part of the deck that's above the 2 with the part that's below the 2; continue in this manner, until finally swapping the part of the deck that's above the N with the part that's below the N. (If one of the "parts" that we swap is empty, the swap is equivalent to simply moving the top card to the bottom or vice versa.)

For example, suppose we start with $1,2,4,3$ (with 1 on the top and 3 on the bottom). We swap the cards above the 1 (all zero of them) with the cards below the 1 (all three of them) to get $2,4,3,1$. Then we swap the cards above the 2 with the cards below the 2 to get $4,3,1,2$. Then we swap the card above the 3 with the cards below the 3 to get $1,2,3,4$ (notice that the 1 and the 2 remain in the order they were in above; though two piles are swapped, each pile stays in its original order). Finally we swap the cards above the 4 with the cards below the 4 to get $4,1,2,3$. So, our grand swap sends $1,2,4,3$ to $4,1,2,3$.

Show that performing $N+1$ successive grand swaps always brings the deck back to its original order.

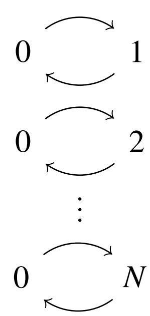
Solution: Add an imaginary card to the deck, numbered 0. We can now represent the order of cards in the deck by a circular arrangement of $0,1,\ldots,N$ (to be read clockwise, let's say), where 0 marks the beginning/end of the deck. For example, the arrangement



would represent a 7-card deck in the order $2,5,1,4,3,7,6$. Note that rotating the circle does not make any difference.

When we swap the part of the deck above k with the part below k, the effect on the circular arrangement is to swap the two arcs delimited by 0 and k. However, because rotations of the circle don't matter, this has the same effect as simply swapping the 0 and the k themselves.

Thus, a grand swap consists of the following N pair swaps:



Tracking each card's label throughout this process, we have

$$\begin{aligned} 0 &\longrightarrow 1 \\ 1 &\longrightarrow 0 \longrightarrow 2 \\ 2 &\longrightarrow 0 \longrightarrow 3 \\ &\vdots \\ N-1 &\longrightarrow 0 \longrightarrow N \\ N &\longrightarrow 0 \end{aligned}$$

Thus, the effect on each number in the circle is to increment it by 1 (with N wrapping around to 0). When we repeat this operation $N+1$ times, every number in the circle cycles back to its original value, and the original order of the deck is restored.

Remark. Throwing in the 0 card and then removing it at the end is an application of the "camel principle" of problem solving. For more about this principle and its widespread uses, see `https://thepalindrome.org/p/the-camel-principle`.