14th Bay Area Mathematical Olympiad

BAMO-8 Exam

February 28, 2012

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The four problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems

A Hugo plays a game: he places a chess piece on the top left square of a 20×20 chessboard and makes 10 moves with it. On each of these 10 moves, he moves the piece either one square horizontally (left or right) or one square vertically (up or down). After the last move, he draws an X on the square that the piece occupies. When Hugo plays the game over and over again, what is the largest possible number of squares that could eventually be marked with an X? Prove that your answer is correct.

B Answer the following two questions and justify your answers:

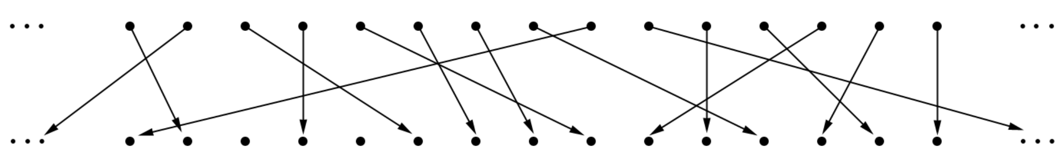
(1) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + 5^{2012}$?

(2) What is the last digit of the sum $1^{2012} + 2^{2012} + 3^{2012} + 4^{2012} + \cdots + 2011^{2012} + 2012^{2012}$?

C Two infinite rows of evenly-spaced dots are aligned as in the figure below. Arrows point from every dot in the top row to some dot in the lower row in such a way that:

- No two arrows point at the same dot.
- No arrow can extend right or left by more than 1006 positions.

Show that at most 2012 dots in the lower row could have no arrow pointing to them.



Please turn over for the remaining problem!

D Laura won the local math olympiad and was awarded a "magical" ruler. With it, she can draw (as usual) lines in the plane, and she can also measure segments and replicate them anywhere in the plane. She can also divide a segment into as many equal parts as she wishes; for instance, she can divide any segment into 17 equal parts. Laura drew a parallelogram $ABCD$ and decided to try out her magical ruler. With it, she found the midpoint M of side CD, and she extended side CB beyond B to point N so that segments CB and BN were equal in length. Unfortunately, her mischievous little brother came along and erased everything on Laura's picture except for points A, M and N. Using Laura's magical ruler, help her reconstruct the original parallelogram $ABCD$: write down the steps that she needs to follow and prove why this will lead to reconstructing the original parallelogram $ABCD$.

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2012 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 11. This event will include lunch, a mathematical talk, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.