# SFMTC season 5, episode 1 — Oct. 30, 2021

## FUN WITH BALLOONS AND PAPER

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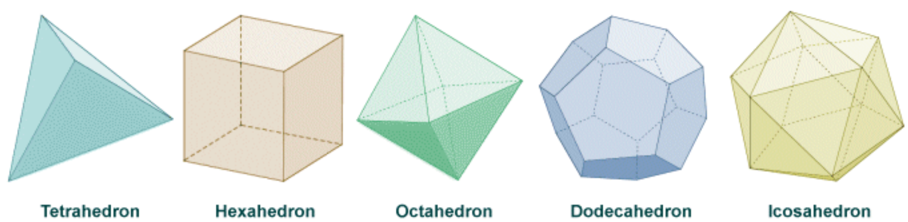
## 1 Balloon polyhedra

Instead of making dogs or hats, we will focus on building polyhedra, specifically, the five platonic solids (hexahedron is a fancy name for a cube).




Below are the "skeletons" of the polyhedra (i.e., the polyhedra as *graphs*):

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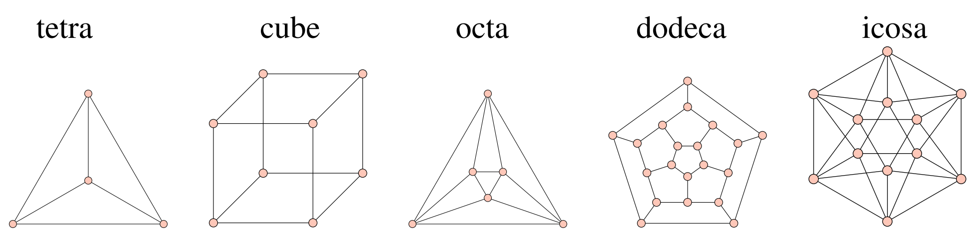
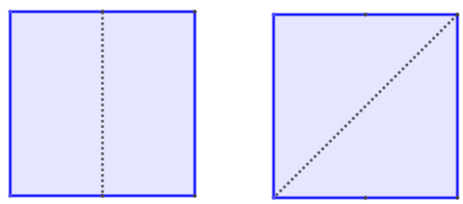


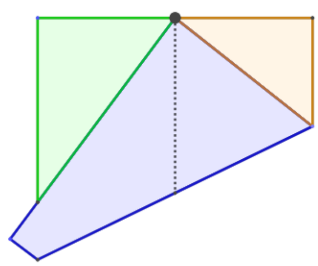

Your assignment is simple: try to make polyhedra using *as few* balloons as possible.

# 2 Doing *math* with a square sheet of paper

Starting with a square sheet of paper, you have no choice but to do one of the two *trivial* folds:



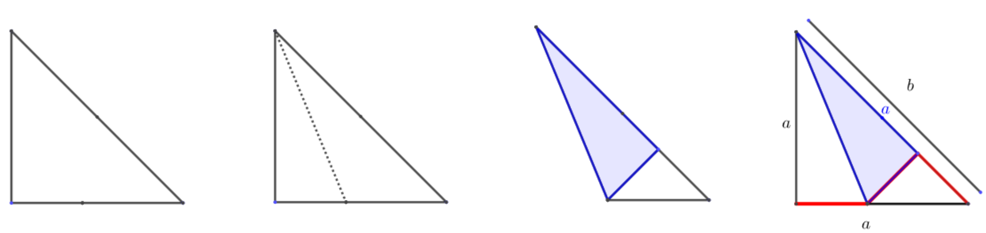
**1** Starting with the vertical trivial fold, make the only *non-trivial* fold:



We claim that this folding produces a wonderful "fraction machine." Try to compute all the different lengths that you now have. Try to do it with as simple math as you can (you can use trig and "hard" algebra, but must you use these tools?)

**2** Starting with a square sheet, can you efficiently make a perfect $1/3$ fold? How about $1/5$? Can you generalize to $1/n$?

**3** If you instead do the diagonal trivial fold, you get an isosceles right triangle. Then do a *non-trivial* fold, and label the lengths.




We claim that this sets the stage for a nice proof for the irrationality of a famous number. Explain!

# SUGGESTIONS (SPOILER ALERTS!)

## 1 Ballon polyhedra

It is fun and kind of silly to build polyhedra out of balloons, but the real purpose of this activity is to get you thinking about *graph theory*, the mathematical study of networks. There are hundreds of good books about this,[1] where you can explore things in much greater depth. For our purposes, we are keeping things very simple.

- A *graph* consists of *vertices* and *edges*, with some vertices connected to other vertices by edges. It may be possible to connect a vertex to itself (a "loop") and one can connect two vertices with more than one edge.
- The *degree* of a vertex is the number of edges emanating from it. This is measured "locally" (from the point of view of a bug living at the vertex), so a vertex with loop has degree two.
- An *Eulerian circuit* is a walk that starts and ends at the same place that visits every edge exactly once. An *Eulerian path* visit every edge exactly once, but starts and ends at different vertices.
- Euler's theorem states that
  - A graph has an Eulerian circuit if and only if all of its vertices have even degree.
  - A graph has an Eulerian path if and only if all but two of its vertices have even degree. The path starts and ends at the two vertices of odd degree.
- **The crux idea is the so-called "bloon theorem" which is a consequence of Euler's theorem.** A *bloon* is defined to be a single balloon, possibly twisted so that it has numerous vertices and edges. The bloon theorem[2] states that
  - If all vertices of a polyhedron are even, then it can be built using a single balloon.
  - If $n$ vertices of a polyhedron are odd ($n > 0$), then the polyhedron can be built using $n/2$ balloons, and no fewer.
- Consequently the only platonic solid that can be built with a single balloon is the octahedron, since the degree of each vertex is 4. The tetrahedron, cube, dodecahedron and icosahedron require, respectively, 2, 4, 10, and 6 balloons!

[1] Two of our favorites are *Pearls of Graph Theory*, by Ringel and Hartsfield, and *Graphs and their Uses* by Ore.

[2] Two very nice papers by Erik Demain, Martin Demain, and Vi Hart "invented" the field of mathematical ballon twisting. "Balloon Polyhedra" (`https://erikdemaine.org/papers/Balloons_ShapingSpace2/paper.pdf`) is easy to read with great illustrations; highly recommended. "Computational balloon twisting" (`http://cccg.ca/proceedings/2008/paper34.pdf`) is only a little more technical. These papers coin the term "bloon" and prove the bloon theorem.

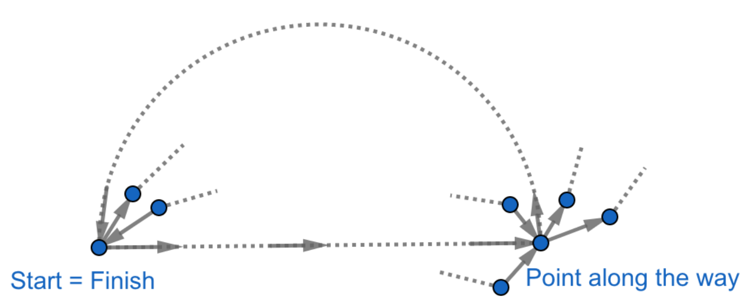
- But the big question is: WHY? First of all, let us take Euler's theorem as a given. Then, for example, if we started at any vertex of an octahedron and walked along the edges, we could perform an Eulerian circuit (starting and ending at the same vertex and visiting each edge exactly once). Clearly, this is a single bloon!

  Now, consider the icosahedron. Each of the 12 vertices has degree three, so the bloon theorem predicts that we need $12/2 = 6$ bloons, no fewer. Consider where the bloon *ends* must be. If we are traveling along the graph, either we stop or start at a vertex (a bloon end) or we travel through. If we travel through, then we are entering and leaving, which uses up two edges. So even-degree vertices cannot be endpoints. By the same reasoning, every odd-degree vertex must be an endpoint. So all 12 of the vertices of the icosahedron are bloon ends. Since each bloon has two endpoints, we will need at least $12/2 = 6$ bloons.

  How do we know that 6 bloons will actually work? We can perform some temporary surgery on the icosahedron, and add 6 new edges. Pick two vertices, and join with a new edge. Then pick two other vertices, and join them, etc. Now each vertex has degree 4, which means that we can do an Eulerian circuit using a single bloon. Now remove the 6 extra edges. We now have six different bloons!

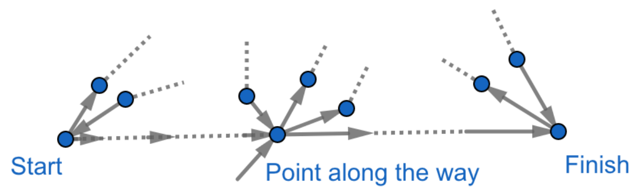
  This idea is completely general. It depends, however, on one subtle fact: that the number of odd-degree vertices must be even. This is a consequence of the *handshake lemma*, which says that in any graph, if you add up the degrees of all the vertices, you must get an even number, namely twice the number of edges. This is easy to see: visit each vertex and count the degree by carving a notch into each edge emanating from that vertex. The sum of the degrees will be the number of notches, and each edge will have two notches (since each edge is shared by two vertices).

- But why is Euler's theorem true? You can easily look this up in the references provided or elsewhere, but it is really pretty simple to see how it works using the same ideas as above. I like to call it the "belly button" proof, because you want to think of walking along the graph and categorizing edges at vertices as "innies" or "outies". Start your walk somewhere. If you have an Eulerian circuit, you end up at the same place, and you visit every edge exactly once.



The number of innies and outies at each vertex must always be equal; hence all degrees are even.

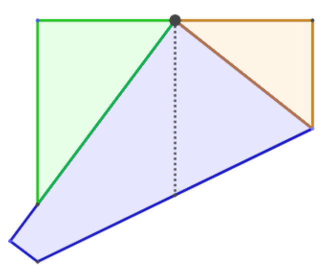
If instead, we have an Eulerian path, then there is a start vertex and and end vertex and these are different.




Each "in-between"vertex must have the same number of innies and outies, but the start vertex must have one more outie than innie, and the finish vertex must have one more innie than outie.

## 2 Folding squares

**1** *The fraction machine*.



The diagram above is very rich; both the green and the brown triangles are 3-4-5 right triangles, for example. And the length of the long leg of the green triangle is $2/3$ of he length of the square. Hence you have found a way to divide a sheet exactly into thirds!

There are many ways to work this out. The key idea *for all folding problems* is that

> *when you fold one point onto another point, the folding crease line is the perpendicular bisector of the line segment joining these two points.*

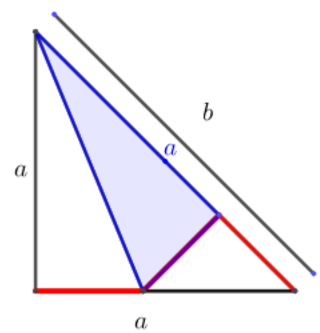
Using this principle, one can employ trig, the Pythagorean theorem, or just simple computations of slopes.

**2** Using this idea, it is a fun challenge (try to keep the algebra simple by thinking about slopes; remember that perpendicular slopes are negative reciprocals of one another) to prove that if the square has a vertical fold $1/n$ of the way along the bottom, then the non-trivial fold will yield a green triangle whose length is $(2n-2)/(2n-1)$. For example, if we have a $1/3$ fold, then the non-trivial fold will yield a $4/5 : 1/5$ division along the left side of the square.

Clearly we can generate all possible fractions! You can look a a simple Geogebra workbook that illustrates this at `https://www.geogebra.org/classic/kd8vfeyh`.

**3** This fold inspires us to prove that $\sqrt{2}$ is irrational. Note that the ratio of hypotenuse to leg in an isosceles right triangle is $\sqrt{2}$.

Suppose, to the contrary, that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = b/a$, where $a, b$ are integers and $b/a$ is in lowest terms. Hence the $a, a, b$ isosceles right triangle is the *smallest* isosceles right triangle in the universe with *integer* lengths. But look at what happened after the non-trivial fold:




The red lengths are all equal to $b-a$, which is an integer. The little triangle at the bottom right is similar to the big $a, a, b$ triangle. But its lengths are $b-a, b-a, a-(b-a)$, and of course these are integers. Smaller integers. Contradiction! This nontrivial folding trick creates smaller and smaller isosceles right triangles. So it is impossible to have a smallest isosceles right triangle with integer sides. So we conclude that $\sqrt{2}$ cannot be rational.