SFMTC **Feb. 24, 2024**

VISUAL, WORDLESS PROOFS

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1 *Recognize these common sequences.*

- Let T_n be the sum of the first n positive integers, often called the nth **triangular number**. The first few values of T_n are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, \ldots$.
- The **squares** are $1, 4, 9, 16, 25, 26, 49, 64, 81, 100, 121, 144, 169, 196, 225 \ldots$.
- The **Fibonacci numbers** f_n are defined by $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

2 *Warmups.*

(a) Find the sum of the first few consecutive odd numbers. Explain and prove.

(b) Find and prove a formula for T_n.

(c) Find the sum of the first few Fibonacci numbers. Explain and prove.

3 *Look for triangular numbers.*

(a) Use this table to experiment and discover some identities involving triangular numbers. And then prove your identities.

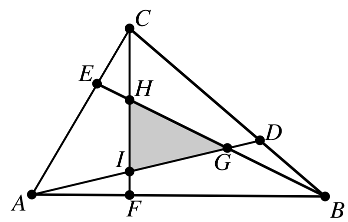
n	T_n	$3T_n$	$8T_n$	$9T_n$
1	1	3	8	9
2	3	9	24	27
3	6	18	48	54
4	10	30	80	90
5	15	45	120	135
6	21	63	168	189
7	28	84	224	252
8	36	108	288	324
9	45	135	360	405
10	55	165	440	495
11	66	198	528	594
12	78	234	624	702
13	91	273	728	819
14	105	315	840	945
15	120	360	960	1080
16	136	408	1088	1224
17	153	459	1224	1377

(b) Let S_n be the sum of the first n squares. For example, $S_4 = 1+4+9+16 = 30$. Likewise, let C_n be the sum of the first n cubes; for example, $C_4 = 1+8+27+64 = 100$. How (and WHY) do S_n and C_n relate to T_n?

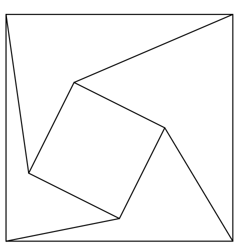
4 Investigate what happens when you square consecutive Fibonacci numbers and add them.

5 Prove the Pythagorean theorem.

6 When you join the vertices of a triangle to a point that divides the opposite side in the ratio 1:2, you get a *threedian*, instead of a median. The threedians do not intersect in a point, but instead create a smaller triangle. What is the ratio of the small triangle to the large mother triangle?



7 *Nested squares*. A large square contains a smaller square which is located anywhere inside with any orientation. Join consecutive vertices of the small square with consecutive vertices of the large square to get the four lines showed in the figure.



Prove that the midpoints of these lines are the vertices of a square!