WHAT (QUILTING) CIRCLES CAN BE SQUARED?

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1. The Problem

Beth received this message a while back:

Hi Beth,

My son, Martin gave me your email address. Hopefully he already told you I'm struggling with organizing a group project for some quilters, and can't seem to get it to work out exactly the way I want it to. Here's the scoop if you feel like a puzzle. I'd certainly appreciate the help and insight.

I am organizing a round robin quilt exchange. That means a group of quilters each make a small quilt block and trade them with each other over a period of time until everyone in the group has had a chance to add a border to everyone else's quilt. On the first trade day, each person passes their quilt block to another person in the group and receives a block from someone else in that group, so they can add a border to that block. After a month or so, the blocks are passed to another person in that group, so they can add a border to the block and so on until everyone in the group has had a chance to sew a border on every block.

The last time we did this activity we passed our project to the same person each time until we got our own quilt back. If I passed my quilt to "Martha" then every time we traded, I gave my quilt to "Martha," which means she always had to follow my work in this game. It is kind of limiting, since part of the fun of doing these trades is getting to meet the other people in the group.

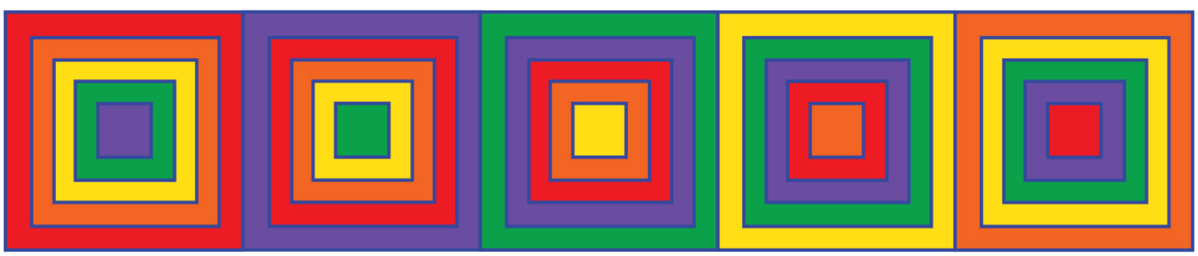

Figure 1. Five quilts created by passing in a cycle. This method was unsatisfactory because each quilter always passed to the same neighbor.

I'd like to arrange the trades so no one ever gets their quilt from the same person, but I can't quite figure it out. I tried it with 5 people in a group and, while everyone gets the quilt to sew on, the quilts are passed twice to the same person.

I think we will pretty much always have 5 people in a group, but it is possible that we would have groups with 4 or 6 people. Is there something that will work for that, too?

Judy

Question 1 (Group Discussion)**.** Can we create a quilt passing pattern for 3 people that works as Judy intends?

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2. Exploration

Question 2. Can we make a quilt passing pattern for 4 people? If yes, demonstrate it. If not, justify why not.

Question 3. Can you solve Judy's problem, and find a quilt passing pattern for 5 people? If yes, demonstrate it. If not, justify why not.

Question 4. Can you describe a general method of passing quilts for an even number of people?

3. RCLS and open questions

Let S be a set of n distinct elements—for example, S may consist of n people, or the integers 0 through $n-1$. An n by n array filled with elements from S such that each element appears once in each row and once in each column of the array is called a *Latin square* of order n.

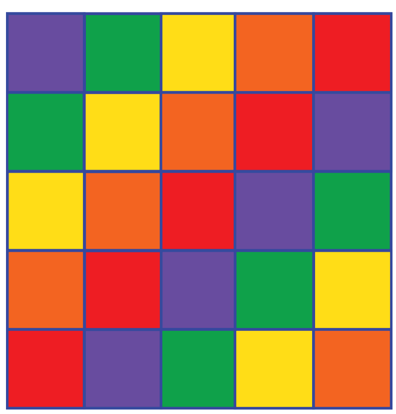

Figure 2. The Latin square of order 5 associated to the quilts in Figure 1.

A round robin quilt exchange can be modeled with a Latin square. Think of each row as the path of a quilt through the quilters, labeled 0 through $n-1$. The condition that no quilt undergoes the same pass twice creates the condition that the sequence (i, j) appears at most once in the rows of the square. There are $n(n-1)$ different sequences (i, j), and $n(n-1)$ quilt passes must take place, so in fact each sequence (i, j) must appear exactly once in the array. An $n \times n$ Latin square with this property is referred to as a *row complete Latin square* of order n, abbreviated RCLS(n). Note that the square in Figure 2 is not row complete. Row completeness was first studied by Williams in 1948, when developing designs for experiments in which treatments might have residual effects, for example, the milk production of dairy cows on various feeds.

In general, it is not known whether RCLS of prime order exist. The first open case is for order 11. (The nonexistence in cases of 3, 5, and 7 have been shown.)

Question 5 (Open Question)**.** Is there a RCLS of order 11?

Question 6 (Major Open Question)**.** Is there a RCLS of order p, for any prime $p > 11$?

4. Rotational RCLS

Each row of an $n \times n$ latin square has a sequence of $n-1$ successive differences. The successive difference corresponding to the sequence (i, j) is $j - i$. We call a Latin square *rotational* if each row has the same sequence of successive differences modulo n.

- In terms of the quilt exchange, this means that there is a single passing pattern that all quilts follow. That is, if the quilters were sitting in a circle, each quilt would pass the same number of steps to the right (for positive numbers) or to the left (for negative numbers) at each stage.
- Williams noticed that the sequence of successive differences $1, -2, 3, -4, 5, \dots, -(n-2), (n-1)$ leads to a rotational RCLS(n) for even n.

Question 7. If all quilts are following the same passing pattern, and every quilt starts at a different person, is it possible that two quilts could "collide" and leave a person with two quilts after a pass? How would you explain why this is possible or prove it is not?

We know that there aren't any RCLS of orders 3, 5, or 7. Using abstract algebra, it's possible to show that there are no rotational RCLS of odd order. So we will focus on even numbers.

Question 8. Try to make up a passing pattern for 4 or 6 people. Does it work? If so, try another! If not, what goes wrong?

Question 9. Let's try a very simple passing pattern. Instead of Williams' back and forth pattern, let's try just passing to the right in larger and larger steps. Try the pattern $1, 2, 3, ..., (n-1)$ for $n = 4, 6, 8, 10, 12, 16$. When does it work? How can you tell? How many rows do you need to check to make sure it works? What goes wrong when it doesn't work? Can you make a conjecture about what n work for this pattern?

Question 10. Let's keep track of how far around the circle a quilt following this pattern has traveled at each stage. We will say that T_i is the total distance a quilt has gone around the circle after each pass. We will keep counting up, even after the quilt passes its starting point. Write down T_0, T_1, T_2 ... etc. Do you see a pattern here/have you seen these numbers before?

5. One Answer and More Questions

In the referenced paper, we prove that T_0 through T_{n-1} are all distinct mod n if and only if n is a power of 2. This proof is totally doable but would take too much time to do it all in this session. So we won't prove that T_0 through T_{n-1} are all distinct mod n if n is a power of 2. Let's assume this is true and talk about the other way!

Here, we get to explore *trapezoidal numbers*–numbers that can be written as the sum of consecutive positive integers (where you have to use at least two numbers–so not every number is trivially trapezoidal. For example, 3 is trapezoidal because $3 = 1 + 2$, and 9 is trapezoidal because $9 = 2 + 3 + 4$, but 2 is not trapezoidal because the only positive integer smaller than 2 is 1 and so no sequence can add up to 2. Trapezoidal numbers form a trapezoid when we draw dots in rows representing the terms in the sum.

Question 11. Thinking about shapes, can you explain why every trapezoidal number is the difference of two triangular numbers?

Question 12. If $T_k - T_j = n$, what can you say about T_k and T_j mod n?

Question 13. Check the numbers 1 through 17 to see which are trapezoidal. Any conjectures on which ones are trapezoidal and which ones aren't?

Dave Roeder, John Watkins, and Carlton Gamer proved that trapezoidal numbers are exactly the non-powers of 2. Bringing the pieces together, we see that this means the consecutive number passing pattern can never work when n is not a power of 2. The motivation for their work was to explore generalizations of 12-tone music, a kind of atonal music that uses every note the same number of times. Carlton Gamer is a well-known composer who used "little pitch cells" (short sequences of successive tones) to make atonal music sound more tonal. He is currently 94 years old and still composing.

Question 14 (Open Question)**.** What other passing patterns work for some categories of n? Are there a finite number of patterns that we will encounter?

6. Sources

(1) Malmskog, Beth, and Kathryn Haymaker. "What (Quilting) Circles Can Be Squared?" *Mathematics Magazine* 92.3 (2019): 173-186.